Summary

Nonexpansive mappings, i.e., 1-Lipschitz, form like isometries one of the basic classes of nonlinear mappings. Many problems of nonlinear analysis can be reduced to studying the dynamics of such mappings. Various classes of nonlinear mappings preserving order on cones in Banach spaces that are nonexpansive with respect to Hilbert's or Thompson's metric have found a number of applications, among others, in game theory, control theory and the nonlinear Perron-Frobenius theory . The relations between nonexpansive mappings and convex analysis are applicable in such disciplines as mechanics, economics, partial differential equations, information theory, approximation theory, signal and image processing, game theory, optimization theory, probability, statistics and computational biology.

The aim of this dissertation is to analyze the asymptotic behavior of the orbits of nonexpansive mappings and their one-parameter semigroups in geodesic spaces. The well-known Banach fixed point theorem describes the behavior of orbits of contraction mappings in complete metric spaces. In other words, if f is a contraction, then there exists exactly one fixed point x_0 of the mapping f such that for every x, a sequence of iterates $f^n(x)$ of f convergences to x_0. Nonexpansive mappings are the limiting case in the theory of contraction mappings, when the Lipschitz constant tends to 1, and their dynamics are much more complicated. In the case of a nonexpansive mapping additional assumptions are necessary to guarantee existence of a fixed point. Moreover, even when there is such a point of mapping f, the sequence of iterates $f^n(x)$ in general does not convergence to a fixed point. The confirmation of complexity of dynamics of nonexpansive mappings is the fact that the study of asymptotic behavior of such mappings is one of the most frequently conducted in nonlinear analysis. Research so far has mostly been performed in Banach spaces, or in metric spaces with nonpositive curvature, so called CAT(0) metric spaces, or in specific hyperbolic metric spaces.

Geodesic spaces, which originally described the geometry of points on the Earth's surface, are metric spaces in which any two points can be connected by a curve of length equal to the distance between these points. In the case of the Earth, the geodesic is a fragment of a great circle. Over time this concept began to be considered more generally, for example in graph theory. Geodesic spaces are an important class of metric spaces, which are of particular importance, for example, in general theory of relativity. Using the properties of these spaces, we will look at the discussed problems of dynamics of nonexpansive mappings in a more general and uniform way.

Thinking about dynamics of nonexpansive mappings it is impossible not to say about the Wolff-Denjoy theorem. In the classical version this theorem has

been formulated and then proved several times in 1926 by J. Wolff and A. Denjoy. The theorem describes the behavior of iterates of holomorphic self-mappings on the unit disc of the complex plane which do not have fixed points.

Theorem
If $f : \Delta \to \Delta$ is a holomorphic mapping of the unit disc $\Delta \subset \mathbb{C}$ without a fixed point, then there is a point $\xi \in \partial\Delta$ such that the iterates f^n of f converge locally uniformly on every compact subset of Δ to ξ.

The Wolff-Denjoy theorem has been generalized over years in different directions, for different kinds of mappings and different types of spaces, both finite and infinite dimensional. Thanks to an in depth analysis of the existing results and the research methods contained in them, it was possible to provide further, new generalizations of the Wolff-Denjoy theorems which are a significant part of this dissertation.

Speaking further about the theory of nonlinear mappings, it is worth mentioning about the Karlsson-Nussnaum conjecture, which is a generalization of the Wolff-Denjoy theorem. This conjecture has been independently formulated by A. Karlsson and R. Nussbaum and it is still an important open problem in this area of research. In this dissertation we will give our counterpart of the Karlsson-Nussbaum conjecture, i.e., our result for resolvents of nonexpansive mappings. Moreover, due to introducing the notion of attractor we will get some particular cases of this conjecture, which are also new results.

Conjecture
Let D be a bounded, convex domain in a real finite dimensional normed space. If $f : D \to D$ is a fixed point free nonexpansive mapping on a Hilbert's metric space (D, d_H), then there exists a convex set $\Omega \subseteq \partial D$ such that for each $x \in D$, all accumulation points $\omega_f(x)$ of the orbit $O_f(x)$ lie in Ω.

Semigroups are important families of mappings. A special case is a one-parameter, continuous semigroup of nonexpansive mappings. Some of its properties are known sa well as a counterpart of the Wolff-Denjoy theorem describing the convergence of the orbits of these semigroups with respect to Kobayashi's distance. Moreover, the existence of a common fixed point for a one-parameter semigroup of nonexpansive mappings with bounded orbits in a uniformly convex Banach space was proved by Browder. In this context, it was also possible to propose a general approach to the Wolff-Denjoy theorem which includes some previous results as special cases. In addition, it is worth noting that in this case we also get results related to the Karlsson-Nussbaum conjecture.

This dissertation consists of six chapters. In the first chapter there are basic definitions, notations, lemmas, notes and examples necessary for a full understanding of this dissertation. We will start with the title nonexpansive mapping and then we will talk about the types of metric spaces which we will be dealing with. We will define the orbit of a mapping and formulate Całka's theorem with its proof, which is one of the classical arguments in this area of research. Next, in chapter one, we will discuss in detail the notions of Hilbert's metric, Thompson's metric, Poincaré's metric and Kobayashi's distance and their important, from the point of view of the dissertation, properties. Following the idea of Beardon and Karlsson, we will formulate special properties of metric spaces, which we will call Axioms. Finally, we will briefly dwell on the definition of horoballs, give their properties and consider whether there is a connection between the concept of a horoball with the previously mentioned axioms.

The second chapter is entirely devoted to a short history about the Wolff-Denjoy theorem. We will give the first version of this theorem proposed in 1926 by J. Wolff. Due to the fact that there have been many generalizations of the Wolff-Denjoy theorem over the years, we are not able to list and discuss all of them. However, we will focus on those that are important from the point of view of this dissertation and those that show the directions of research on this problem.

The third chapter will be devoted to our new results concerning generalizations of the Wolff-Denjoy theorem in proper geodesic spaces. It will be divided into two main parts, i.e., the discrete case concerning dynamics of iterates of nonexpansive mappings and the continuous case concerning one-parameter continuous semigroups of such mappings. In both cases, at the beginning we will formulate general results with the proofs, and then conclusions for bounded, strictly convex domains in real or complex finite dimensional vector spaces. It is worth adding that in the continuous case we will present a counterpart of Całka's theorem, which seems to be a new result, unheard of in the literature.

The fourth chapter will be about the Wolff-Denjoy theorem in an infinite dimensional case. The considerations of this chapter are an independent contribution of the author of this dissertation. It will be necessary to introduce (λ, κ)-quasi geodesic mertic spaces, a notion of compact mapping and a slight modification of the axioms given in the first chapter. Similarly as in the third part of this dissertation, we will divide our results into two main parts, i.e., discrete and continuous case. Moreover, in both subchapters we will propose a counterparts of the Wolff-Denjoy theorem for bounded, strictly convex domains in Banach spaces.

The fifth chapter will be devoted to considerations on the second research problem, i.e., the Karlsson-Nussbaum conjecture. At the beginning, we will briefly describe what this conjecture is all about. This problem is still open but we know some specific solutions to it, which will also appear at the beginning of this chapter. Next, we introduce the concept of an attractor and with its help we formulate and prove theorems that can be called special variations on the Karlsson-Nussbaum conjecture. Moreover, using similar techniques, we will give a much shorter proof of the theorem that the attractor of a nonexpansive mapping with respect to Hilbert's metric is contained in a star-shaped subset of the boundary of the space. In the fourth subchapter, we will discuss the concept of attractor in the context of one-parameter semigroups and we will formulate the counterparts of theorems from the discrete case. At the end of the fifth chapter, we will construct the resolvent of a nonexpansive mapping, next we will prove its necessary properties and finally we give our result being a counterpart of the discussed conjecture for such mappings.

In the last, sixth chapter, we will recall the concept of the Gromov hyperbolic metric space and give some of its properties. Next, we will discuss some characterization of geodesic boundary in CAT(κ) metric spaces with negative curvature. Using the Wolff-Denjoy type theorem we will provide a slightly simplified proof of the theorem, which shows the relation between geodesic boundary and the fixed point property.