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Foundations of geometry for secondary schools and prospective teachers

(doctoral thesis)

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Contents

Introduction

(0) This thesis is dedicated to the basics of plane synthetic geometry and concerns the congruence of triangles, parallel lines, and similar figures. Firstly, it originates from the observation that while synthetic geometry is a vital part of high school curricula, academic courses for prospective teachers provide mere information on the synthetic approach enumerating axioms, preferably of the Hilbert system. Secondly, secondary school textbooks misrepresent the foundations of geometry and hardly employ the benefits of IT technologies. With this thesis, we aim to revise the curriculum for high schools and propose a course on synthetic geometry for prospective mathematics teachers. Concerning the theory of similar figures, we go beyond the standard axiomatic approach and spell out a primer of automatic theorem proving. Synthetic geometry is 2,500 years old yet accepts up-to-date.

(1) The thesis has five chapters. Chapter 1 includes a synthetic geometry course designed for prospective teachers of mathematics, governed by the following basic premise: a system of education allows various textbooks and pupils have access to multiple channels of information. Accordingly, a teacher should master many axiomatic systems rather than a specific one.

The Hilbert system of axioms makes the core of our course. It provides a basis for reviewing Euclid's *Elements* and a reference in presenting recent treatments of the Euclidean geometry developed by Borsuk, Szmielew, Tarski, and Hartshorne.

Due to our focus on the secondary school curriculum, sections 1.1.2—1.1.3 contain results about congruent triangles, parallel lines, and concurrent lines in a triangle. In these sections, we follow Hartshorne's approach presented in (Hartshorne 2000, ch. 2). He retrofits the foundations of geometry with Hilbert's axioms to bring the treatment up to modern standards of rigor. His academic proficiency merges with the experience he got lecturing on Euclidean and non-Euclidean geometry for undergraduate students (Hartshorne 2000).

(Hartshorne 2000, §11, 12) interpreters Euclid's *Elements* propositions Book I to IV. He proves the existence of an isosceles triangle, enabling him to mirror Euclid's constructions and spells out the Pasch axiom and its follow-ups, such as the concept of a half-plane and the crossbar theorem, to fill up some gaps in the Euclid system. These parts of Hartshorne's treaty prove helpful in comparing Euclid and Hilbert's systems.

In section 1.1.4, we present a version of Thales' theorem for commensurable line segments being an introduction to the general version detailed in Chapters 2 to 4. The moral of this section is: while the commensurable version of Thales' theorem appeals to elementary techniques, the general version requires arithmetic of line segments or real numbers. Our position, advanced in Chapter 3, refers to Euclid's proof of the Thales theorem (VI.2) and opens a perspective on automatic theorem proving.

Section 1.2 contains a thorough discussion of Book I of the *Elements* based on the English translation (Fitzpatrick 2007), especially the so-called absolute geometry (I.1—28), parallel lines and parallelograms (I.29—34), and the Pythagorean theorem (I.46—48). A detailed presentation of propositions aims to expound gaps in Euclid's arguments and explain modern developments of Euclidean geometry, specifically the role of the Pasch and circle-circle axiom. In that section, we adopt Hartshorne's idea of the construction tools to compare the Hilbert and Euclid systems in developing the absolute geometry. On the one hand, these consist of copying line segments and angles, on the other, the use of Euclidean straightedge and compass. From the perspective of construction tools, we also review modern refinements of the Hilbert system, namely (Borsuk, Szmielew 1972) and (Tarski 1959). Furthermore, we refer to constructions while introducing automatic theorem proving: next to introducing points (constructions stage), it applies a reverse process of elimination. In this way, we link the automated theorem proving with 2500 years old technique of Greek geometry.

In section 1.4, we present systems of Euclidean geometry, developed by Borsuk and Szmielew in (Borsuk, Szmielew 1972) and Tarski in (Tarski 1959). With that part, we aim to show that these systems, Hilbert's system and Euclid's Book I of the *Elements*, share the same pattern: an absolute geometry (*Elements*, I.1–28) followed by the theory of parallel lines (*Elements*, I.29 and on). Based on these comparisons, our recommendation for the school curriculum is as follows: criteria for congruent triangles should precede propositions about parallel lines. Moreover, the results about congruent triangles should be taught in secondary schools as a collection of truths, even though SAS is the axiom and SSS, ASA, and AAS are theorems.

We can also draw some observations concerning construction tools based on comparing various systems. Thus, instead of Euclidean straight edge and compass, (Hilbert 1899) adopts a straight edge and rigid compass, i.e., transportation of line segments. (Borsuk, Szmielew 1972) and (Tarski 1959) adopt the copying of line segments and refine Hilbert's approach by replacing line segments with equal distances between pair of points, the copying of an angle with a triangle construction, and SAS with the five-segment axiom. We tally these results in a table, summarizing section 1.4, and refer back to it when reviewing school textbooks in Chapter 5.

Every course in synthetic geometry must discuss the Parallel Postulate (PP) and demonstrate its independence, i.e., that it does not follow from the axioms of the absolute geometry. In the celebrated proposition I.32, Euclid shows that PP implies that angles in any triangle sum up to π. Hartshorne shows (Hartshorne 2000, 321–322), that the reverse implication obtains in Archimedean planes. Max Dehn, in a 1900's paper (Dehn 1900), presented a model of the absolute geometry where PP is not satisfied, yet angles in triangles sum up to pi; it got the name of semi-Euclidean plane. In section 1.5, we present another model of semi-Euclidean plane. It is a subspace of the non-Archimedean plane over the field of hyperreal numbers $\mathbb{R}^*$. In our model, PP is not satisfied, angles in triangles sum up to pi, but in contrast to Dehn's model, the circle-circle axiom is satisfied, as well as the standard (modern) Euclidean trigonometry.

Usually, hyperbolic planes are models of absolute geometry which do not satisfy PP. While these models change Euclid's concept of a straight line or angle, in our model, straight lines and angles are Euclidean. Last but not least, our model has a unique educational advantage: expounding its crucial ideas requires only the basics of Cartesian geometry and non-Archimedean fields.

(2) Thales', or intercept, theorem, occasionally named the fundamental theorem of proportionality, plays a crucial role in our project, as it is a breaking-through topic that differs between Euclid's approach and modern axiomatic systems. In the *Elements*, that theorem is a part of Book VI that builds on the proportion developed in Book V. While proportion was the basic technique of ancient Greeks, 20th-century geometry does not use it anymore, and mainstream mathematics replaced proportion with arithmetic of real numbers (Błaszczyk 2021).

In the *Elements*, Thales' theorem is merely proposition VI.2 relying on VI.1 – the only one in Book VI that refers back to the definition of proportion (V. def. 5). The Euclidean proportion is a relation between pairs of figures of the same kind, line segments being of one kind, triangles of another, and angles of yet another. That concept includes that line segments, angles, and triangles are comparable in terms of greater-than. Modern systems define a total order of line segments and angles, but do not consider any order of triangles; that is why they cannot apply ancient proportions to triangles. Thales' theorem, thus, goes beyond standard techniques described in Chapter 1 and requires novel tools. Through the 20th century, the arithmetic of line segments and real numbers addressed the problem.

Chapter 2 provides an overview of the proofs of Thales' theorem in the 20th-century systems of geometry. We specifically discuss those developed in (Hilbert 1899), (Hartshorne 2000), (Birkhoff 1932), (Millman, Parker 1991), (Borsuk, Szmielew 1960), and (Schwabhäuser, Szmielew, Tarski 1983). Each of these systems approves our claim that Thales's theorem is quite a challenge. We identified two general strategies in that regard: segment arithmetic (Hilbert 1899, Hartshorne 2000, Schwabhäuser, Szmielew, Tarski 1983), and real numbers being implemented into the system of geometry (Borsuk, Szmielew 1960, Millman, Parker 1991). The latter strategy has two variants: a system includes axioms that guarantee the existence of bijection between real numbers and a straight line (Birkhoff 1932) or derives that bijection from the Hilbert-style axioms (Borsuk, Szmielew 1960).

Through sections 2.1–2.2, we review proofs of Thales theorem applying arithmetic of line segments. In 2.3 – Borsuk and Szmielew's theorem on the existence of measure, i.e., a bijection mapping real numbers on a straight line; in 2.5 – Millman and Parker's proof of the Thales' theorem. Section 2.4 contains the Birkhoff axiom on the relationship between real numbers and a straight line.

(3) Modern systems of geometry seeking to bypass Euclid's proof of Thales' theorem introduce real numbers or the line segments arithmetic. The resulting proofs are involved and hardly match school textbooks. Euclid's proof is as simple as it can be, yet it employs the technique cast out by modern mathematics. Teachers try to copy it but, finally, emulate Euclid's wording only: instead of a proportion, they use real numbers. We propose an in-between solution: an axiomatic theory that makes proposition VI.1 an axiom and enables one to recover its original proof; it also enriches our course with the 21st-century techniques of automated theorem proving.

Chapters 3 and 4 develop a new approach to Thales' theorem, based on the axioms of the area method and the associated automatic theorem proving in the GCLC system developed in (Chou, Gao, Zhang 1994) and (Janičić, Narboux, Quaresma 2012). Throughout these chapters, we consider similar figures and school tasks related to proportions.

In section 3.1, we present the axiomatic area method after (Janičić, Narboux, Quaresma 2012) and then, in section 3.3, show that the system is consistent. To this end, we provide a model: Cartesian plane $\mathbb{R} \times \mathbb{R}$ with a lexicographic order that enables us to interpret the concept of the directed segment and signed area. In section 3.2, we demonstrate how to render Euclid's proof of proposition VI.2 without relying on Greek proportion.

The so-called co-side theorem is the fundamental one of the area method. In section 3.4, we prove it and show how to use it in school mathematics. In addition to Euclid's VI.1 exploring proportions of triangles with the same height, the co-side theorem considers triangles on the same basis. Four graphic patterns summarize geometrical insights comprised in these theorems. Under our recommendation, they should be a part of school mathematics related to proportion.

(4) In Chapter 4, we proceed from the axiomatic treatment of the area method to an automatic theorem proving. One aspect of that approach concerns a cognitive paradigm shift: understanding arguments in synthetic geometry consists of capturing axioms and rules of inferences, in an automatic theorem proving – capturing elimination of points. To elaborate on this: the co-side theorem enables one to replace triangles occurring in terms, representing a theorem with line segments, e.g., a ratio of two-line segments can replace a ratio of two triangles. Elimination lemmas specify this procedure. Given that, an automated proof proceeds as follows:

1. Te thesis of a theorem is translated into an expression in the area method language,

2. Given some starting points, new points are introduced, one by one, through the allowed constructions (construction stage),

3. Each point introduced in the construction stage is eliminated based on elimination lemmas, but in reverse order, i.e., the last constructed is the first in the elimination process, etc. (elimination stage),

4. The process reaches identity $1 = 1$ and stops.

To put it into a bigger perspective: straight edge and compass constructions define ancient Greek geometry, constructions allowed in the area method and elimination of points – automated proofs through the area method.

A brief introduction to the GCLC automatic theorem proving program (section 4.2) precedes automatic proofs of propositions from Book VI of Euclid's *Elements* (section 4.3). That part of our thesis is an addendum to Euclid's proofs presented in (Błaszczyk, Mrówka 2013). These proofs, even when translated into the language of the GCLC program, are readable to a layman unfamiliar with the programming vocabulary. Interestingly, the proposed method enables one to reconstruct Euclid's theses and his proof technique, i.e., the theory of proportions.

Euclid's *Elements* Book VI contains the Greek theory of similar figures: criteria for similar triangles (VI.4–7), areas of similar figures (VI.19, 20), or the so-called generalized Pythagorean theorem (VI.30). Since secondary school curricula include modern counterparts of these propositions, our recommendation for teachers' courses is to introduce both Euclid's and automatic proofs. Finally, one can take a bigger perspective and compare the contribution of Chapter 4 with the standard treatments of Thales' theorem and similar triangles detailed in Chapter 2. Our approach combines the axiomatic method and current IT technology in providing a reconstruction of Euclid's arguments, but it also assumes the standard Euclidean geometry. It presents Euclid's geometry and the theory of similar triangles as branches of mathematics rather than a system based on unique foundations. In that respect, it differs from the standard synthetic geometries presented in Chapter 2.

(5) In Chapter 5, we review secondary school textbooks presentations of elementary geometry from the perspective of the four following criteria:

Whether a textbook includes a discussion concerning the concepts of point and line;

Whether it discusses concepts related to the Pasch axiom like a side of a line, a half-plane, the separation of a plane, or the cross-bar theorem;

Whether it introduces algebra of line segments and angles or processes their measures;

Whether criteria for congruent triangles precede the parallel axiom or a textbook puts them in reverse order.

We tally the results of that review in a table corresponding to a table in section 1.4, which compares the 20th-century classics in synthetic geometry authored by Hilbert, Borsuk, Szmielew, and Tarski.

We also review these textbooks from the perspective on how they introduce Thales' theorem.

Section 5.3 includes a review of university courses on elementary geometry.

Section *Conclusions* details our recommendations concerning the secondary school curriculum and university courses on synthetic geometry. Chapter 1 of our thesis is a course on synthetic geometry for prospective teachers. We recommend enriching it with the automatic theorem proving regarding Thales' theorem and related propositions, as presented in Chapter 4. A recommendation for the secondary school curriculum includes, i.a., an introduction to the application *euclidea* and novel graphic patterns combining Euclid's proposition VI.1 and the co-side theorem.

Chapter 1

Plane geometry

1.1 Plane geometry by Hartshorne

1.1.1 Hilbert's axioms

Hilbert *Grundlagen der Geometrie*, from (Hilbert 1899) to (Hilbert 1972) got eleven editions. Hartshorne's (Hartshorne 2000) includes its modern version adjusted to educational practice. Hilbert axioms, as presented therein, differ from the original only in applying modern symbols.[1] Here they are, grouped by Hilbert according to primitive concepts of his system: point, straight line, the relation of betweenness, congruence of line segments, and angles.

Hilbert system of axioms (by Harsthorne)

Axioms of Incidence

I1. For any two distinct points A, B, there exists a unique line l containing A, B.

I2. Every line contains at least two points.

I3. There exist three noncollinear points (that is, three points not all contained in a single line).

Axioms of Betweenness

B1. If B is between A and C, (written $A * B * C$), then A, B, C are three distinct points on a line, and also $C * B * A$.

[1] (Greenberg 2008, 597–602), provides a concise account of Hilbert axioms, while (Hartshorne 2000) introduces them in pace as theory develops.

B2. For any two distinct points A, B, there exist points C, D, E such that $A * B * C$, $A * D * B$, and $E * A * B$.[2]

B3. Given three distinct points on a line, one and only one of them is between the other two.

B4. (Pasch). Let A, B, C be three non collinear points, and let l be a line not containing any of A, B, C. If l contains a point D lying between A and B, then it must also contain either a point lying between A and C or a point lying between B and C, but not both.

Axioms of Congruence for Line Segments

C1. Given a line segment AB, and given a ray r originating at a point C, there exists a unique point D on the ray r such that $AB \cong CD$.

C2. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.

C3. (Addition). Given three points A, B, C on a line satisfying $A*B*C$, and three further points D, E, F on a line satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

Axioms of congruence for Angles

C4. Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$, on a given side of the line DF, such that $\angle BAC \cong \angle EDF$.

CS. For any three angles α, β, γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

C6. (SAS) Given triangles ABC and DEF, suppose that $AB \cong DE$ and $AC \cong DF$, and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent, namely, $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.

Archimedes' axiom (A)

Given line segments AB and CD, there is a natural number n such that n copies of AB added together will be greater than CD.

Parallel axiom (P)

For each point A and each line l, there is at most one line containing A that is parallel to l.

[2]Hartshorne adopts the existence of C and shows how to prove the existence of D and E (Hartshorne 2000, 73–79); Hilbert adopts the existence of C and D (Hilbert 1899, 82); Greenberg – C, D, and E (Greenberg 2008, 597). The third version explicitly states that points on a line are *dense*, as well as the line can be *prolonged to the right* or *left*.

Absolute geometry is a part of classical geometry without the parallel axiom, i.e., Axioms of Incidence+Axioms of Betweenness+ Axioms of Congruence for Line Segments+ Axioms of Congruence for Angles.

In elementary geometry on a plane we identify groups of "important" theorems. The first such group consists of theorems about congruent triangle. The second group, theorems about intersections of "special" lines in a triangle (of bisectors, side bisectors, altitudes, medians). The third group concerns theorem about similar triangles.

1.1.2 Congruence of Triangles

Triangle congruence theorems are theorems of absolute geometry and they are a very important element of elementary geometry. We are starting with Hilbert's axioms and are introducing only the necessary theorems to prove tringle congruence theorems.

Note that when we speak about the Hilbert plane, we mean a set of points and lines on which the axioms of incidence, of betweenness and congruence are satisfied. Geometry on the Hilbert plane is the absolute geometry. The second remark is that SAS is a Hilbert's axiom C6 and so it needs no proof. We also do not show proofs of the theorems about addition and subtraction of segments and angles. We know that it results directly from Hilbert's axioms.

In (Hartshorne 2000), § 7, Hartshorne introduces the concept *side of line l*. It is an equivalence relation between points of plane not lying on l defined by: $A \sim B$ iff $A = B$ or segment AB does not meet l. It determines two equivalence class, called sides of l, or half-planes.

Since there are three noncollinear points, relation $\sim$ determines at least one equivalence class. Hartshorne shows there are at most two classes:

(1) If A, B, C are not collinear, we consider the triangle ABC (see. Fig1.1 left). From $A \not\sim C$ we conclude that $\overline{A}C$ meets l. From $B \not\sim C$ we conclude that BC meets l. Now by Pasch's axiom it follows that $\overline{A}B$ does not meet l. So $A \sim B$ as required.

(2) Suppose A, B, C lie on a line m (see. Fig1.1 right). Choose a point D on l, not on m, and use (B2) to get a point E with $D * A * E$. Then $A \sim E$ as we showed above.

Now, $A \not\sim C$ by hypothesis, and $A \sim E$, so we conclude that $C \not\sim E$, since $\sim$ is an equivalence relation (if $C \sim E$, then $A \sim C$ by transitivity: contradiction).

Theorem 1.1.1. *Let $\angle BAC$ be an angle, and let D be a point in the interior of the angle.*

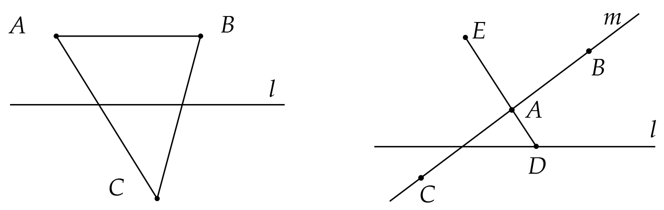


Figure 1.1: there are at most two equivalence classes of sides of l

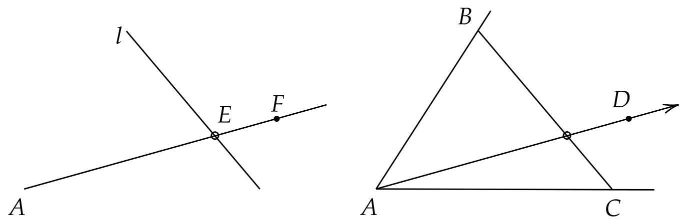


Figure 1.2: Finding points on different sides of l (left) and crossbar theorem (right)

Then the ray $AD^{\rightarrow}$ must meet the segment BG.

Proof. All points of the segment EB, except B, are on the same side of $\overrightarrow{AB}$ (denote l). By the construction $E * A * C$, this mean all points of the segment EB, except B, are on the opposite side of l from C. D is in the interior of the $\angle BAC$, all the points of the ray AD, except A, are on the same side of l as C. Thus the segment BE does not meet the ray AD.

All points of the segment EB, except E, lie on the same side of $\overleftrightarrow{AC}$ (denote m) as B, while the points of the ray of n, opposite the ray $\overrightarrow{AD}$, lie on the other side of m. Hence, the segment BE cannot meet the ray opposite to $\overrightarrow{AD}$. Together with the previous step, this shows that the segment BE does not meet the line $\overleftrightarrow{AD}$. We conclude (using the Pasch's axiom for the $\triangle EBC$) that $\overrightarrow{AD}$ must intersect BC, since it does not intersect with BE.

B and F are on the same side of m, and also B and D are on the same side of m, so D and F are on the same side of m, and so D and F are on the same side of A on the line n. In other words, F lies on the ray AD.

$\square$

Theorem 1.1.2. *For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.*

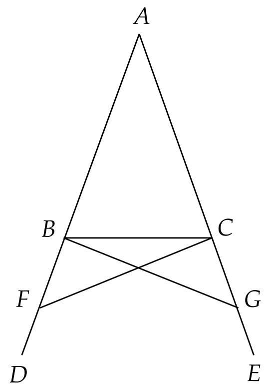


Figure 1.3

Proof. Let ABC be the given isosceles triangle, with $AB \cong AC$. Points A, B defind line $\overleftrightarrow{AB}$, points A, C defind line $\overleftrightarrow{AC}$. We choose the point F on the line AB. We put point G on the ray $\overrightarrow{AC}$ so that $AF \cong AG$.

$\triangle ABG \cong \triangle ACF$ (SAS): $AB \cong AC$, $AF \cong AG$, $\angle BAC$ is common. This means that $BG \cong CF$, $\angle ABC \cong \angle ACF$, $\angle AFC \cong \angle AGB$.

$BF \cong CG$ because $AB \cong AC$, $AF \cong AG$.

$\triangle FCB \cong \triangle GBC$ (SAS): $BF \cong CG$, $BG \cong FC$, $\angle AFC \cong \angle AGB$. This means that $\angle FBC \cong \angle GCB$.

$\angle ABG \cong \angle ACF$ and $\angle CBG \cong \angle FCG$ since $\angle ABC \cong \angle ACB$ □

Theorem 1.1.3 (SSS). *If two triangles ABC and $A'B'C'$ have their respective sides equal, namely $AB \cong A'B'$, $AC \cong A'C'$, and $BC \cong B'C'$, then the two triangles are congruent.*

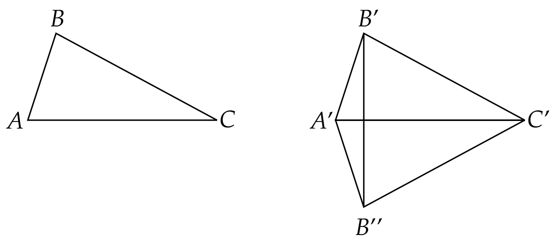


Figure 1.4

Proof. Construct $\angle C'A'B'' \cong \angle CAB$ on the other side of the ray $A'C'$ from B' and make $A'B'' \cong AB$.

$\triangle A'C'B'' \cong \triangle ABC$ (SAS): $A'B'' \cong AB$, $AC \cong A'C'$, $\angle C'A'B'' \cong \angle CAB$. This means that $BC \cong B''C$. If $A'B' \cong AB$ and $AB \cong A'B''$, then $A'B' \cong A'B''$, thus triangle $A'B'B''$ is isosceles and $\angle A'B''B'\angle \cong \angle A'B'B''$. Similarly, triangle $B'B''C'$ is isosceles and $\angle C'B''B'\angle \cong \angle C'B'B''$. By addition of congruent angles $\angle A'B''B' + \angle C'B''B'\angle \cong \angle A'B'B'' + \angle C'B'B''$. We can conclude that $\angle A'B'C' \cong \angle ABC$ and $\triangle ABC \cong \triangle A'B'C'$. □

Theorem 1.1.4 (ASA)**.** *If two triangles ABC and $A'B'C'$ have their respective angles equal, namely $\angle BAC \cong \angle B'A'C'$, $\angle BCA \cong \angle B'C'A'$, and $AC \cong A'C'$, then the two triangles are congruent.*

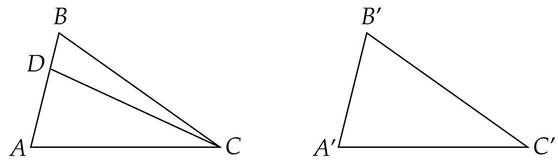


Figure 1.5

Proof. Let AB be uncongruent $A'B'$ and let $AB > A'B'$. Construct the point D on the line AB so that $AD \cong A'B'$ and $A * D * B$. $\triangle ADC \cong \triangle A'B'C'$ (SAS): $AD \cong A'B'$, $AC \cong A'C'$, $\angle DAC \cong \angle B'A'C'$. This means that $\angle ACD \cong \angle A'C'B'$ and $DC = B'C'$.

We have $\angle BCA \cong \angle B'C'A'$ and $\angle ACD \cong \angle A'C'B'$, but there exist a unique ray on a given side of the given line, which is a contradiction, and $AB = A'B'$ □

Theorem 1.1.5. *If $\angle BAC$ and $\angle BAD$ are supplementary angles, and if $\angle B'A'C'$ and $\angle B'A'D'$ are supplementary angles, and if $\angle BAC \cong \angle B'A'C'$, then also $\angle BAD \cong \angle B'A'D'$.*

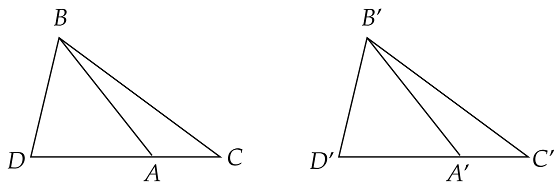


Figure 1.6

Proof. We can always choose points A', B', C', D' on rays such that $AB \cong A'B'$, $AC \cong A'C'$ and $AD \cong A'D'$.

$\triangle BAC \cong \triangle B'A'C'$ (SAS): $AB \cong A'B'$, $AC \cong A'C'$, $\angle BAC \cong \angle B'A'C'$. This means that $\angle BCD \cong \angle B'C'D'$ and $BC = B'C'$.

$DA + AC = DC$, $D'A' + A'C' = D'C'$ and $AC \cong A'C'$, $AD \cong A'D'$ since $DC \cong D'C'$

$\triangle DBC \cong \triangle D'B'C'$ (SAS): $DC \cong D'C'$, $BC \cong B'C'$, $\angle BCD \cong \angle B'C'D'$. This means that $\angle BDC \cong \angle B'D'C'$ and $BD = B'D'$.

$\triangle DBA \cong \triangle D'B'A'$ (SAS): $DA \cong D'A'$, $DB \cong D'B'$, $\angle BDA \cong \angle B'D'A'$. Hence, we have $\angle BAD \cong \angle B'A'D'$. $\square$

Theorem 1.1.6. *Vertical angles are congruent.*

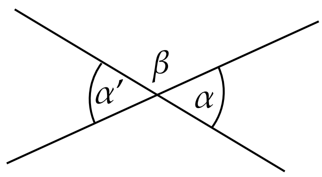


Figure 1.7

Proof. The vertical angles α and α' are each supplementary to $\angle\beta$, and $\angle\beta$ is congruent to itself, so by the Theorem 1.1.5, $\angle\alpha \cong \angle\alpha'$. $\square$

Theorem 1.1.7. *Given a line segment AB, there exists an isosceles triangle with base AB.*

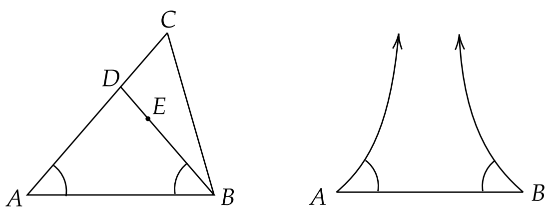


Figure 1.8

Proof. Let AB be the given line segment. Let C be any point not on the line $\overleftrightarrow{AB}$. If $\angle CAB \cong \angle CBA$, then $\triangle ABC$ is isosceles. If not, then one angle is less than the other. Let $\angle CAB < \angle CAB$. Construct $\angle ABE \cong \angle CAB$ such that a ray $\overrightarrow{BE}$ in the interior of the $\angle CBA$.

Now we have to show that $\overrightarrow{AC}$ and $\overrightarrow{BE}$ meet, only the equality of the angles is not enough (see right Figure 1.8). By the crossbar theorem, $\overline{BE}$ must meet the opposite side AC in a point D. Now we have $\angle DBA \cong \angle DAB$, so $\triangle ABD$ is isosceles. $\square$

Theorem 1.1.8. *There is a midpoint for each segment.*

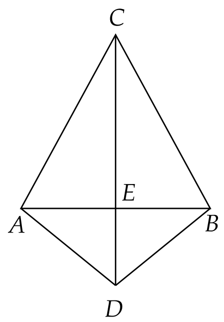


Figure 1.9

Proof. Given a line segment AB, there exists an isosceles triangles ABC and ABD so that points C and D lie on the opposite sides of the line AB. Draw the segment CD. Hence, $\triangle ACD \cong \triangle BCD$ by SSS. Now we have $\triangle ACE \cong \triangle BCE$. This means that $AE \cong EB$. $\square$

Theorem 1.1.9. *In any triangle, the exterior angle is greater than either of the opposite interior angles.*

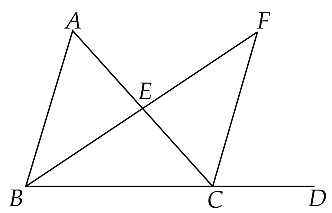


Figure 1.10

Proof. ABC be a given triangle. Choise point D on the ray BC such that $B * C * D$. $\angle ACD$ is an exterior angle of the $\triangle ABC$. We show $\angle ACD > \angle BAC$.

Let E is point on AC such that $AE \cong EC$, and let F is point on AE such that $BE \cong EF$. $\angle AEB \cong \angle FEC$ as a vertical angle. So, we have $\triangle AEB \cong \triangle CEF$ and this means $\angle BAE \cong \angle ACF$.

Since $B * E * F$ and $B * C * D$, points F and D are on the same side of the line AC. On the other hand, $A * E * C$ and $B * E * F$, thus points A and F are on the same side of the line BD. P is in the interior of the $\angle ACD$ and ray CF also. So $\angle DCA > \angle FCA$, we conclude from it $\angle DCA > \angle BAE$, □

Theorem 1.1.10. *If two triangles ABC and $A'B'C'$ have their respective angles equal, namely $\angle BAC \cong \angle B'A'C'$, $\angle BCA \cong \angle B'C'A'$, and $AB \cong A'B'$, then the two triangles are congruent.*

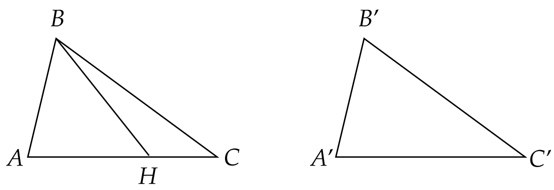


Figure 1.11

Proof. Let AC be unequal $A'C'$ and let $AC > A'C'$. Construct the point H on the line AC so that $AH \cong A'C'$ and $A * H * C$. $\triangle ABH \cong \triangle A'B'C'$ (SAS): $AH \cong A'C'$, $AB \cong A'B'$, $\angle BAH \cong \angle B'A'C'$. This means that $\angle BHA \cong \angle B'C'A'$ and $BH = B'C'$. Thus, $\triangle HBC$ is isosceles and $\angle BHC \cong \angle BCH$.

$\angle BHA$ is an exterior angle of $\triangle BHC$. By the Theorem 1.1.9 $\angle BHA > \angle BHC$, but on the other hand $\angle BHC = \angle BHA$, which is a contradiction. We conclude that $AC \cong A'C'$ and $\triangle ABC \cong \triangle A'B'C'$. □

Theorem 1.1.11. *For every line l and every point P, there exist a unique line m such that P is on m and $m \perp l$.*

Proof. We choose points Q and Q' on the line l. We construct $\angle PQP'$. Next we construct $\angle Q'QP'$ on the other side of the ray QQ' from P that is congruent to $\angle PQQ'$, and make QP' congruent to QP. Let F be intersection of PP' and l.

$\triangle QFP \cong \triangle QFP'$ (SAS): $PQ \cong PQ'$, QF is common, $\angle PQF \cong \angle P'QF$. That means that $\angle PFQ \cong \angle P'FQ$. Hence, $\angle PFQ$, $\angle P'FQ$ are right angles and $l \perp PP'$. Now we have to show that its perpendicular line is unique. Let there be another line PR perpendicular to l passing through the point P. Then $\angle PFQ$ is the exterior angle of $\triangle FRP$ and, by the

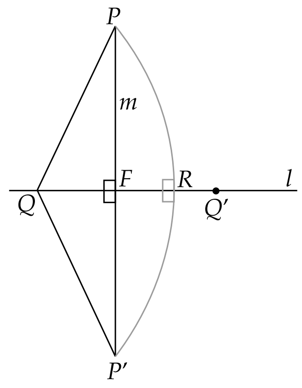


Figure 1.12

Theorem 1.1.9, $\angle PFQ > \angle PRF$, but on the other hand $\angle PFQ = \angle PRF$ as right angles, which is a contradiction. We conclude that PP' is a unique line perpendicular to l, passing through the point P. $\square$

Theorem 1.1.12. *If the hypotenuse and a leg (HL) of a right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the triangles are congruent.*

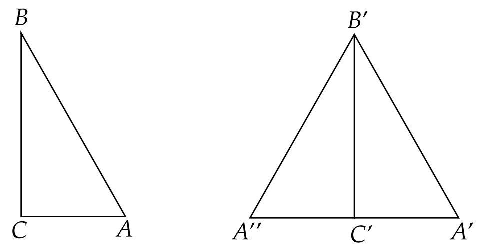


Figure 1.13

Proof. We have two right triangles ABC and $A'B'C'$ such that $\angle C$, $\angle C'$ are right, $AB \cong A'B'$, $BC \cong B'C'$. Construct a point A'' on line $A'C'$, on the side of C' opposite A', so that $A''C \cong CA$.

$\triangle A''C'B' \cong \triangle ABC$ (SAS): $BC \cong B'C'$, $AC \cong A''C'$, $\angle BCA \cong \angle B'C'A''$. Then $BA \cong B'A''$ and, on the other hand, $BA \cong B'A'$ since $B'A' \cong B'A''$.

But the triangle $A''A'B'$ is an isosceles triangle, and $B'C'$ is an altitude since $\triangle C'A'B' \cong \triangle C'A''B'$. Since the latter triangle is congruent to $\triangle CAB$, then $\triangle C'A'B' \cong \triangle CAB$.

$\square$

1.1.3 Concurrent Lines in a Triangle

Theorem 1.1.13. *A circle can be inscribed into any triangle. Its center is the point of intersection of the bisectors of this triangle.*

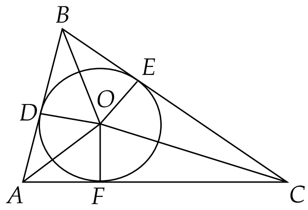


Figure 1.14

Proof. Let ABC be a given triangle, and point O be the intersection point of the bisectors, drawn from vertices A and C. Let us show that BO is the bisector of $\angle ABC$. Let $OF \perp AC$, $OE \perp BC$, $OD \perp BA$ (by Theorem 1.1.11).

$\triangle AFO \cong \triangle$ (AAS): AO is common, $\angle DAO \cong \angle FAO$, $\angle ODA \cong \angle OFA$. This means that $DO \cong OF$. It can be shown analogously $OF \cong OE$, since $DO \cong OE$.

$\triangle BDO \cong \triangle BOE$ (HL): $DO \cong OE$, BO is common. Since $\angle DBO \cong \angle EBO$, this means BO is a bisector of $\angle ABC$.

Finally $DO \cong OE \cong OF$ and we conclude that points D, E, F lie on the same circle centered at the point O.

$\square$

Theorem 1.1.14. *The three perpendicular bisectors of the sides of a triangle meet in a single point.*

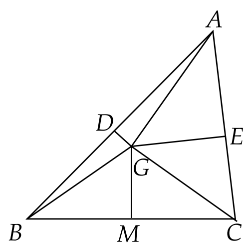


Figure 1.15

Proof. Let ABC be a given triangle, $BM \cong MC$, $CE \cong EA$, $AD \cong DB$, $MG \perp BC$, $EG \perp AC$.

$\triangle BGC$ is isosceles: $BM \cong MC$, GM is an altitude. Then $BG \cong GC$. Analogously, we can show that $GC \cong GA$. Hence, we have $BG \cong GA$. It means $\triangle BGA$ is isoscales. DG is a median in $\triangle ABC$. From the property of an isosceles triangle we conclude that DG is an altitude. Hence, DG is a perpendicular bisector of the side BA.

□

Theorem 1.1.15. *The three altitudes of a triangle meet in a single point (the orthocenter of the triangle).*

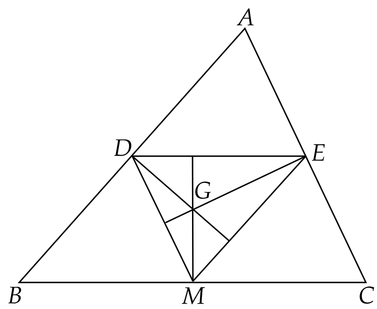


Figure 1.16

Proof. Let DME be the given triangle. Draw lines through the vertices D, M, E, parallel to the opposite sides, to form a new triangleABC. $BMED$ and $MCED$ are parallelograms we see that $BM = DE = MC$. Thus M is the midpoint of BC, and similarly for the other two sides of $\triangle ABC$ we have D is the midpoint of BA, E is the midpoint of AC.

On the other hand, the altitude GM of the $\triangle MDE$ is perpendicular to DE, and, hence, also perpendicular to BC. Thus, we see that the altitudes of the $\triangle DME$ are equal to the perpendicular bisectors of the sides of the $\triangle ABC$. Hence, they meet in a single point by the Theorem 1.1.14.

□

Theorem 1.1.16. *Let ABC be a triangle, and let D, E be the midpoints of AB and BC, respectively. Then the line DE is parallel to the base AC, and equal to one-half of it. In other words, if F is the midpoint of AC, then $DE \cong AF$. DE is called midline in $\triangle ABC$*

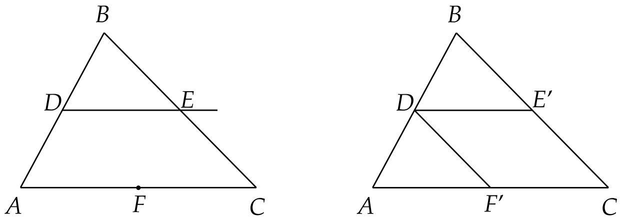


Figure 1.17

Proof. Draw lines through D parallel to AC and BG. Let them meet the opposite A sides in points E', F'. Since $DF' \parallel BC$ and $AC \parallel DE'$, $\angle DBE' \cong \angle ADF'$ and $\angle F'AD \cong \angle E'DB$. $\triangle ADF' \cong \triangle DBE'$ (ASA). We conclude that $AF' \cong DE'$ and $DF' \cong BE'$. $F'DE'C$ is a parallelogram, it means $DE' \cong F'C$ and $DF' \cong E'C$. Thus, we see that E' and F' are the midpoints of the sides BC and AC. So $E' = E$, $\overleftrightarrow{DE'} = \overleftrightarrow{DE}$, and therefore DE is parallel to AC as claimed. Furthermore, we have seen that $DE' \cong AF'$, and F' is the midpoint of AC, so DE is equal to one-half of AC.

□

Theorem 1.1.17. *Let ABC be a triangle, and let D_1, E_1 be points on AB and BC, respectively, so that $AB = nBD_1$ and $BC = nBE_1$. Then the line D_1E_1 is parallel to the base AC, and $D_1E_1 = \frac{1}{n}AC$.*

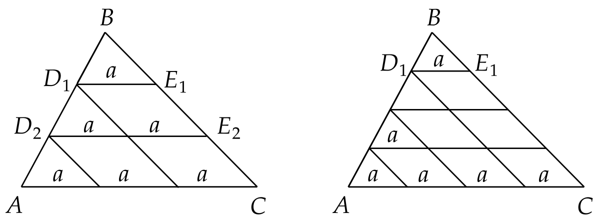


Figure 1.18

Proof. Let's divide AB and BC into n equal parts. Let $D_1E_1 = a$. Through the point D_1 draw a line parallel to BC. D_1E_1 is midline in $\triangle D_2BE_2$, this means $D_2E_2 = 2a$ and $D_2E_2 \parallel D_1E_1$. On the next step through the point D_2 draw a line parallel to BC. We get triangle with base parallel to D_1E_1 and equal $2a$ and one parallelogram with base a. Analogically, on the step $n-1$ we get triangle with base parallel to D_1E_1 and equal $2a$ and $n-2$ parallelograms with base a (see Fig. 1.18). Hence $AC \parallel D_1E_1$ and $AC = nD_1E_1$.

□

Note: the inverse is also true if $AC \parallel D_1E_1$ and $AC = nD_1E_1$, then $AB = nBD_1$ and $BC = nBE_1$. The way of the proof is analogous, but starts by dividing the AC into n equal parts.

Definition 1.1.18. We say a triangle ABC is n-congruent to the triangle FED, in symbols $ABC \cong nFED$, if the three sides of ABC are n-sides of FED, and the three angles of ABC are equal to the three angles of FED.

For example on the Figure 1.18 on the left $\triangle ABC \cong 3\triangle D_1BE_1$ and on the right $\triangle ABC \cong 4\triangle D_1BE_1$.

Theorem 1.1.19. *Let ABC and $A'B'C'$ be two triangles, and assume that the angles at B and C are equal to the angles at B' and C', and that $BC \cong nB'C'$. Then the $\triangle ABC$ is n-congruent to the $\triangle A'B'C'$.*

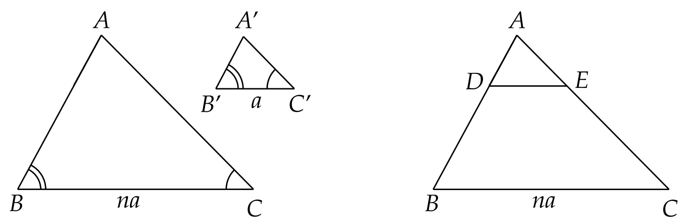


Figure 1.19

Proof. Draw DE parallel to BC so that $AD \cong A'B'$. Then $\triangle ADC \cong \triangle A'B'C'$ (ASA). Now $BC \parallel DE$ and $BC = nDE$, then $AB = nAD$ and $AC = nAE$. This means that the $\triangle ABC$ is n-congruent to the $\triangle A'B'C'$. □

Theorem 1.1.20. *The medians of a triangle meet in a single point (called the centroid of the triangle).*

Proof. Let ABC be the triangle, let D, E be the midpoints of AB and AC, and draw DE. Let the two medians BE and CD meet at a point G. Since DE is parallel to BC, we find that $\angle DEG = \angle CBG$ and $\angle EDG = \angle BCG$. On the other hand, $BC = 2DE$. Therefore, we can apply the previous result and find that $\triangle BGC \cong \triangle 2EGD$.

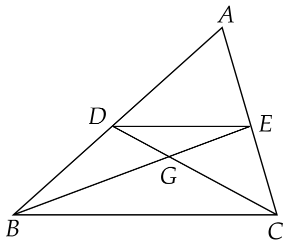


Figure 1.20

In particular, $BG = 2GE$. Thus G can be described as the point on the median BE that is $\frac{2}{3}$ of the way from B to E. Reversing the roles of A and C would therefore show that the third median AF also passes through G. Thus, all three medians meet in the point G.

□

1.1.4 Thales' (intercept) theorem for commensurable line segments

Theorem 1.1.21. *Let $\frac{BD}{DA} = \frac{m}{n}$, where $m, n \in \mathbb{N}$. DE is parallel to one of the sides BC of triangle ABC if and only if BD is to DA, so CE is to EA.*

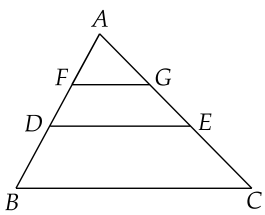


Figure 1.21

Proof. 1) Let $DE \parallel BC$. We have to show $\frac{BD}{DA} = \frac{EC}{AE}$

By the assumption $\frac{BD}{DA} = \frac{m}{n}$ or otherwise $nBD = mDA$.

Let F lie on AB, so that $AF = \frac{1}{m+n}AB$. Draw lines through F parallel to BC. Let it meet the opposite F sides in the point G.

$\triangle ABC \cong (m+n)\triangle AFG$: $\angle AGF = \angle ACB$, $\angle AFG = \angle ABC$, $AB = (m+n)AF$).

Hence, $AC = (m+n)AG$.

$\triangle ADE \cong n\triangle AFG$: $\angle ADE = \angle ABC$, $\angle AED = \angle ACB$, $AD = nAF$.

Hence, $AE = nAG$.

$EC = AC - AE = (m+n)AG - nAG = mAG$.

Finally, we have $\frac{EC}{AE} = \frac{mAG}{nAG} = \frac{m}{n}$

2) Let $\frac{BD}{DA} = \frac{EC}{AE} = \frac{m}{n}$. We have to show $DE \parallel BC$.

Analogously, draw FG, so that $(m+n)AF = AB$ and $(m+n)AG = AC$. This means $\triangle ABC \cong (m+n)\triangle AFG$. Hence, $FG \parallel BC$.

From the other side $nAF = AD$ and $nAG = AE$. This means $\triangle ADE \cong n\triangle AFG$. Hence, $FG \parallel DE$.

Now we have $FG \parallel BC$ and $FG \parallel DE$, finally $DE \parallel BC$.

□

1.1.5 Absolute geometry based on Hilbert's system

Theorem 1.1.22. *Two lines in a plane either have one point in common or none at all. Two planes have no point in common, or have one line and otherwise no other point in comnon. A plane and a line that does not lie in it either have one point in common or none at all.*

Theorem 1.1.23. *Through a line and a point that does not lie on it, as well as through two distinct lines with one point in common, there always exists one and only one plane.*

Theorem 1.1.24. *For two points A and C there always exists at least one point D on the line AC that lies between A and C.*

Theorem 1.1.25. *Of any three points A, B, C on a line there always is one that lies between the other two.*

Theorem 1.1.26. *Given any four points on a line, it is always possible to label them A, B, C, D in such a way that the point labeled B lies between A and C and also between A and D, and furthermore, that the point labeled C lies between A and D and also between B and D.*

Theorem 1.1.27. *Between any two points on a line there exists an infinite number of points.*

Theorem 1.1.28. *Every line a that lies in a plane α: separates the points which are not on the plane α: into two regions with the following property: Every point A of one region determines with every point B of the other region a segment AB on which there lies a point of the line a. However any two points A and A' of one and the same region determine a segment AA' that contains no point of a.*

Theorem 1.1.29. *Every plane α separates the other points of space into two regions with the following property: Every point A of one region determines with every point B of the other region a segment AB on which there lies a point of α; whereas, two points A and A' of one and the same region always determine a segment AA' that contains no point of α.*

Theorem 1.1.30. *In a triangle the angles opposite two congruent sides are congruent, or briefly, the base angles of an isosceles triangle are equal.*

Theorem 1.1.31 (first congruence theorem for triangles)**.** *A triangle ABC is congruent to a triangle $A'B'C'$ whenever the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle A \equiv A'$ hold.*

Theorem 1.1.32 (second congruent theorem for triangles)**.** *A triangle ABC is congruent to another triangle $A'B'C'$ whenever the congruences $AB \equiv A'B'$, $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$ hold.*

Theorem 1.1.33. *If an angle $\angle ABC$ is congruent to another angle $\angle A'B'C'$ then its supplementary $\angle CBD$ is congruent to the supplementary angle $\angle C'B'D'$ of the other angle.*

Theorem 1.1.34. *Let h, k, l and h', k', l' be rays emanating from O and O' in the planes α and α', respectively. Let h, k and h', k' lie simultaneously on the same or on different sides of l and l', respectively. If the congruences $\angle(h, l) \equiv \angle(h', k')$ and $\angle(k, l) \equiv \angle(k', l')$ are satisfied then so is the congruence $\angle(h, k) \equiv \angle(h', k')$.*

Theorem 1.1.35. *Let the angle $\angle(h, k)$ in the plane αa be congruent to the angle $\angle(h', k')$ in the plane α', and let l be a ray in the plane α that emanates from the vertex of the angle $\alpha(h, k)$ and which lies in the interior of this angle . Then there always exists one and only one ray l' in the plane α' that emanates from the vertex of the $\angle(h', k')$ and which lies in the interior of this angle in such a way that $\angle(h, l) \equiv \angle(h', l')$ and $\angle(k, l) \equiv \angle(k', l')$.*

Theorem 1.1.36. *If two points Z_1 and Z_2 are placed on different sides of a line XY and if the congruences $XZ_1 \equiv XZ_2$ and $YZ_1 \equiv YZ_2$ hold, then the angle $\angle XYZ_1$ is congruent to the angle $\angle XYZ_2$.*

Theorem 1.1.37 (third congruence theorem for triangles)**.** *If in two triangles ABC and $A'B'C'$ each pair of corresponding sides is congruent then so are the triangles.*

Theorem 1.1.38. *If two angles $\angle(h', k')$ and $\angle(h'', k'')$ are congruent to a third angle $\angle(h, k)$ then the angle $\angle(h', k')$ is also congruent to angle $\angle(h'', k'')$.*

Theorem 1.1.39. *Let any two angles $\angle(h,k)$ and $\angle(h',l')$ be given. If the construction of $\angle(h,k)$ on h' on the side of l' yields an interior ray k' then the construction of $\angle(h',l')$ on h on the side of k yields an exterior ray l, and conversely.*

Theorem 1.1.40. *All right angles are congruent to each other.*

Theorem 1.1.41 (theorem of the exterior angle)**.** *The exterior angle of a triangle is greater than any interior angle that is not adjacent to it.*

Theorem 1.1.42. *In every triangle the greater angle lies opposite the greater side.*

Theorem 1.1.43. *A triangle with two equal angles is isosceles.*

Theorem 1.1.44. *Two triangles ABC and $A'B'C'$ are congruent to each other if the congruences $AB \equiv A'B'$, $\angle A \equiv \angle A'$ and $\angle C \equiv \angle C'$ are satisfied.*

Theorem 1.1.45. *Every segment can be bisected.*

1.2 Review of Book I of the *Elements*

Hartshorne (Hartshorne 2000, 102) introduces the term *Hilbert construction tools*, meaning transportation (copying) of line segments and angles. Hilbert axioms C1 and C4 decree these tools and also state the uniqueness of respective line segments and angles – the stipulation, uncommon in the *Elements* (proposition I.7 is the only exception), plays a key role in Hilbert-style demonstrations.

Hilbert tools can be reduced to the first (Hartshorne 2000, 185–186). A tool enabling the copying of line segments is called a divider, or gauge, or rigid compass (Martin 1998, Pambuccian 1998, Beeson 2008). Hilbertean constructions, thus, are accomplished with the use of straightedge and divider. Euclidean constructions use a straightedge and compass. In proposition I.3, Euclid shows how to copy line segments. Therefore Euclidean constructions seem more effective than Hilbertean. We adopt a perspective of construction tools to contrast Euclid and Hilbert approaches and seek to identify a Euclidean construction that can not be accomplished with a straightedge and divider.

Hilbert construction tools require a grown theory to justify constructions. In the *Elements*, on the contrary, a theory develops step by step with new constructions, meaning they constitute a deductive structure of the system. And indeed, whereas *Postulates* 1–3 introduce straightedge and compass, *Postulate* 5 is the famous parallels axiom. Thus, from the perspective of the *Elements*, construction tools and the parallels axiom belong to the same category of basic rules.

1.2.1 Transportation of line segments. I 1–3

I.1 *To construct an equilateral triangle on the given line AB.*

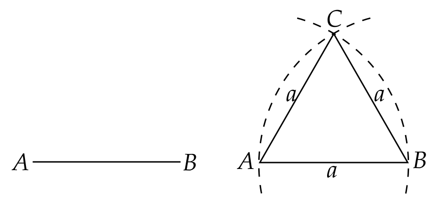


Figure 1.22: *Elements*, I.1 – schematized.

Given that a stands for the line-segment AB, point C, the third vertex of the wanted triangle is an intersection of circles (A, a) and (B, a), i.e., circles with centers at A, and B, and radius equal a.

In tables like the one below, we lay out points resulting from intersections of straight lines and circles.[3]

$$\frac{(A, a), (B, a)}{C}$$

Obviously, there are two solutions to that problem, yet at that stage, there are no means in the Euclid system to show theses two triangles are equal.

In the sequel, we use the following abbreviations explained one after another while going through the subsequent propositions of Book I of the *Elements*.

I.2 *To place a straight-line at point A equal to the given straight-line BC.*

On the line-segment AB, we construct an equilateral triangle ABD with side a; the accompanying diagram depicts it in gray (its *shadow*). Point G is the intersection of the circle (B, b)

[3] The idea of such tables originates from (Martin 1998).

$AB^{\rightarrow}$	extension of line segment AB	*Postulate* 2
(A, a)	circle with center A and radius a	*Postulate* 3
Ab	transportation of line segment b to point A	I.2
mid AB	midpoint of line segment AB	I.10
$AB \perp C$	perpendicular to line AB through point C	I.11, 12
$AB\alpha$	transportation of angle α to line segment AB at point A	I.23
$A \parallel BC$	parallel to BC through point A	I.27, 31
sq on AB	square on line segment AB	I.46

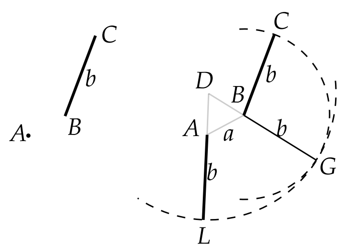


Figure 1.23: *Elements*, I.2 – schematized.

and the half-line $DA^{\rightarrow}$ – the extension of the line segment DB according to *Postulate* 2. Now, DG represents the sum of line-segments a, b. Circle $(D, a+b)$ intersects the half-line $DB^{\rightarrow}$ at point L. Due to the *Common Notions* 3, AL proves to be equal b.

$(B, b),\ DB^{\rightarrow}$	$(D, a+b),\ DA^{\rightarrow}$
G	L

Owning to I.1–2, b is placed at A in a very specific position. Drawing a circle (A, b), one can choose any other position at will, and that is the substance of proposition I.3.

I.3 *To cut off a straight-line equal to the lesser C from the greater AB.*

At first, line-segment b is transported to A into position AL; the accompanying diagram depicts the *shadow* of that construction; let Ab be its symbolic representation. The intersection of the circle (A, b) and the line-segment AB determines E such that $AE = b$.

Summing up, due to I.1–3, one can transport any line segment to any point and position. An equilateral triangle is a tool to this end, while the existence of circle-circle and circle-line

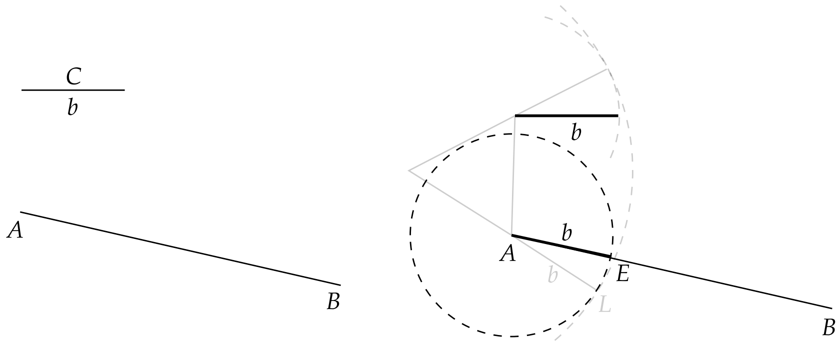


Figure 1.24: *Elements*, I.3 – schematized.

$$\begin{array}{r|c} Ab & (A,b),\ AB \\ \hline & E \end{array}$$

intersection points are taken for granted.

The Euclid system requires a circle-circle or circle-line axiom, both finding grounds in *Postulates* 1–3 that introduce straight-edge and compass. Logically, these two tools reduce to compass alone (*vide* Mohr-Mascheroni theorem), yet, throughout the ages, the economy of diagrams prevailed, and no one questioned the rationale for Euclid instruments. There are, however, models of the Hilbert system that do not satisfy the circle-circle axiom (Hartshorne 2000, 168); (Martin 1998, 91). Hartshorne shows (Hartshorne 2000, 147), that the counterpart of Euclid proposition I.22 (construction of a triangle out of the given sides) is not universally carried out in the Hilbert system. Moreover, in absolute geometry, it is not true that there exists an equilateral triangle with a given length (Pambuccian 1998). Hartshorne shows the existence of the isosceles triangle (Hartshorne 2000, 100), which it plays the role of equilateral triangle in Euclid's propositions such as I.9–12. Anyway, already at the very first propositions of the *Elements*, we observe that Euclid and Hilbert's systems follow alternative deductive tracks. These facts indicate that one cannot simply merge Hilbert's axioms with Euclid's arguments.

Yet another set of problems relates to an intersection of two lines. We will address that question in a commentary to proposition I.10, discussing the concept *side of line*.

1.2.2 Congruence of triangles: SAS to SSS. I 4–8

Throughout propositions I.1–34, *equality* means congruence, whether applied to line segments, angles, or triangles. In I.5–8, Euclid pursues to show the SSS theorem (side-side-side congruence rule), then assumes I.4, *Common Notions*, and characteristics of the *greater-than* relation. In the Hilbert system, I.4 is axiom C6, addition and substruction of *things* referred to in Common Notions are defined, and respective relations proved, similarly with relation *greater-than* between line segments and angles. However, in the *Elements*, *greater-than* refers to all *magnitudes*, i.e., line segments, angles, triangles, figures, and solids. The 20th-century versions of elementary geometry introduce concepts of measure (area, content, etc.) to cover that part of Euclid's geometry. In Book 1, the *greater than* between line segments, angles, and triangles plays a crucial role in propositions I.6–I.32, whereas starting with I.33, Euclid develops the theory of equal figures, crowned by the squaring of a polygon in II.14.

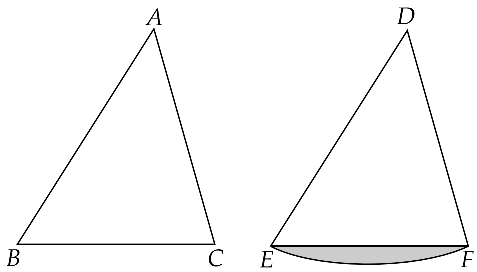


Figure 1.25: *Elements*, I.4 – grey area added.

I.4 *If two triangles have two corresponding sides equal, and have the angles enclosed by the equal sides equal, then they will also have equal bases, and the two triangles will be equal.*

The proof of I.4 (SAS criterion) relies on the *ad hoc* rule: *two straight-lines can not encompass an area.* Figure 1.25 depicts an area encircled by the base EF of the triangle and a curve with ends E, F. By contrast, Hilbert axioms guarantee a unique line through points E, F and there is no room for diagram such as Fig. 1.25 in the Hilbert system.

Since Hilbert proved other axioms of his system do not imply I.4, there is no need to ponder Euclid's argument. Yet, it is worth mentioning that the 20th-century courses of Euclidean geometry, especially ones dedicated to secondary schools, still apply the method of superposition.

I.5 *Let ABC be an isosceles triangle. I say that the angle ABC is equal to ACB.*

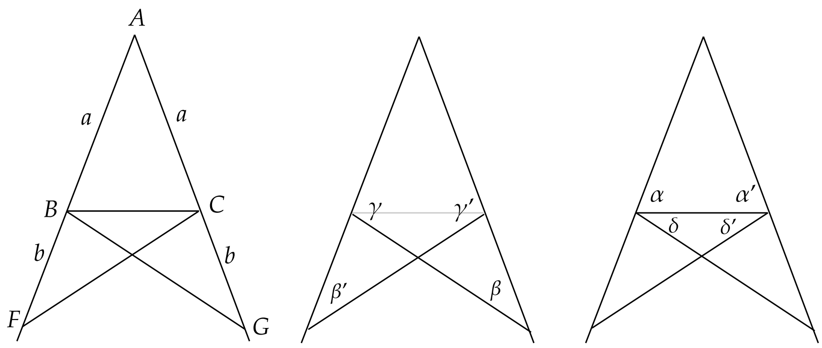


Figure 1.26: *Elements* I.5 – scheme of the proof

The construction part is simple: F is taken at random on the half-line $AB^{\rightarrow}$, then G such that $AF = AG$ is determined on the half-line $AC^{\rightarrow}$.

$$\begin{array}{c|c} AB^{\rightarrow} & (A, a+b),\ AC^{\rightarrow} \\ \hline F & G \end{array}$$

Now, due to SAS, $\triangle GAB = \triangle FAC$. Thus $FC = BG$ and

$$\beta = \angle AGB = \angle AFC = \beta',$$

$$\gamma = \angle ABG = \angle ACF = \gamma'.$$

Again by SAS, $\triangle BFC = \triangle BGC$, and

$$\delta = \angle CBG = \angle BCF = \delta'.$$

By CN 3, $\gamma - \delta = \gamma' - \delta'$. Since

$$\alpha = \gamma - \delta, \quad \gamma' - \delta' = \alpha',$$

the equality $\alpha = \alpha'$ holds. □

I.6 *Let ABC be a triangle having the angle ABC equal to the angle ACB. I say that side AB is also equal to side AC.*

The proof reveals assumptions in no way conveyed through definitions or axioms. At first, it is the trichotomy law for line segments. Let $AB = b$, $AC = c$ (see Fig. 1.27). To reach a contradiction Euclid takes: if $b \neq c$, then $b < c$ or $b > c$. Tacitly he assumes that exactly one of the conditions holds

$$b < c, \quad b = c, \quad b > c.$$

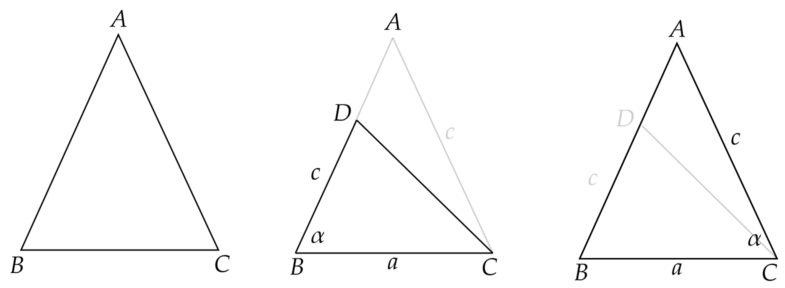


Figure 1.27: *Elements* I.6 – scheme of the proof.

Let $b > c$. Then the construction follows: "let DB, equal to the lesser AC, have been cut off from the greater AB". However, given that angles at B and C are equal, then $AB = c$, and the cutting off "the lesser AC from the greater AB" cannot be carried out. On the other hand, if $AB = b$ and $b > c$, the triangle ABC is not isosceles, and angles at B, C are not equal. Throughout the proof, thus, the diagram *changes* its metrical characteristics and cannot meet the assumptions of the proposition. Contrary to Euclid's claim, D is a random point on AB, rather than introduced *via* a construction captured in the following table

$$\frac{(B, c), AB}{D}$$

Now, by SAS, the equality of triangles $\triangle DBC = \triangle ACB$ holds, and Euclid concludes *the lesser to the greater. The very notion is absurd.*

This time, the trichotomy law applies to triangles. The contradiction

$$\triangle DBC = \triangle ACB \ \ \& \ \ \triangle DBC < \triangle ACB$$

occurs, given the tacit rule: *For triangles, exactly one of the following conditions holds*

$$\triangle_1 < \triangle_2, \quad \triangle_1 = \triangle_2, \quad \triangle_1 > \triangle_2.$$

I.7 *On the segment-line AB, two segment lines cannot meet at a different point on the same side of AB.*

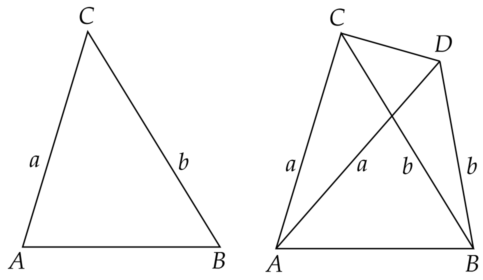


Figure 1.28: *Elements*, I.7 – letters a, b added

The proof, atypically, includes no construction. To get a contradiction, Euclid assumes there are two points C, D such that $AC = a = AD$ and $BC = b = BD$ (see Fig. 1.28).

Both triangles $\triangle ACD$ and $\triangle BCD$ are isosceles and share the common base CD. In the first, angles at the base are equal, $\alpha = \alpha'$. Similarly, in the second triangle, $\beta = \beta'$ (see Fig. 1.29).

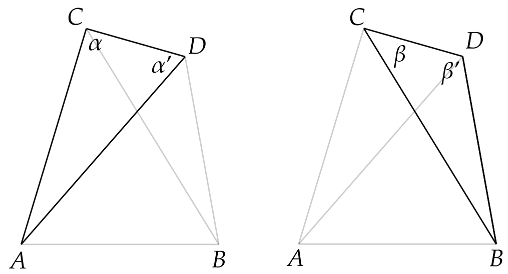


Figure 1.29: *Elements*, I.7 – scheme of the proof

At the vertex C, the inequality $\alpha > \beta$ is visualized, while at D, $\beta' > \alpha'$.[4] Thus, $\beta' > \beta$ and, as stated earlier, $\beta' = \beta$. *The very thing is impossible* – clearly, because exactly one of the conditions holds

$$\beta' < \beta, \quad \beta' = \beta, \quad \beta' > \beta.$$

That proof assumes the trichotomy rule for angles and transitivity of *greater-than* relation. By modern standards, it is, thus, a total order.[5]

[4](Błaszczyk, Mrówka, Petiurenko 2020) expounds the term *visual evidence* in a bigger context. (Beeson, Narboux, Wiedijk 2019), p. 216, identifies Euclid's reliance on the *greater-than* relation and suggests in I.7 one should consider the dimension of the space.

[5]Euclid applies the phrase "is much greater than" when referring to the transitivity.

In I.8, Euclid literally states the SSS criterion. Since the proof relies on a superposition of triangles, we propose the following paraphrase:

If two triangles share a common side and have other corresponding sides equal, then their corresponding angles will also be equal.

In I.9–12, it is employed in that form as Euclid considers two equal triangles on both sides of the common side; in I.23, it is employed to copy angles, yet, the construction of perpendicular plus SAS would do to that end.

Proof of that modification of I.8 effectively reduces to I.7.[6] Similarly, it does not include a construction part, meaning point G is only postulated rather than introduced through straightedge and compass (see Fig. 1.30).

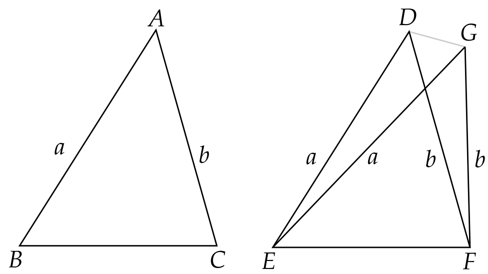


Figure 1.30: Proof of I.8 schematized

From I.8 on, Euclid considers two congruent triangles on both sides of a shared base.

1.2.3 *Greater-than* and *Common Notions*

Through §§ 10–11 of (Hartshorne 2000), Hartshorne seeks to prove Euclid's propositions I.1–34 within the Hilbert system, except I.1 and I.22, as they rely on the circle-circle axiom. He observes that "Euclid's definitions, postulates, and common notions have been replaced by the undefined notions, definitions, and axioms" in the Hilbert system. Commenting on Euclid's proof of I.5–8, Hartshorne writes: "Proposition I.5 and its proof is ok as they stand. [...] every step of Euclid's proof can be justified in a straightforward manner within the framework of a Hilbert plane. [...] Looking at I.6 [...] we have not defined the notion of inequality of triangles. However, a very slight change will give a satisfactory proof. [...] I.7 [...] needs some

[6]Proof of I.7 gets complicated when point D lies inside triangle ABC.

additional justification [...] which can be supplied from our axioms of betweenness [...]. For I.8, (SSS), we will need a new proof, since Euclid's method of superposition cannot be justified from our axioms" (Hartshorne 2000, 97–99).

The above comparison between Euclid's and Hilbert's axiomatic approach simplifies rather than expounds. Euclid implicitly adopts *greater-than* relation between line segments, angles, and triangles as primitive concepts; similarly to addition and subtraction (a *lesser* from the *greater*). In the previous section, we have shown that he takes transitivity and the trichotomy law to be self-evident. One can recover further characteristics from his theory of magnitudes developed in Book V – the only part of Euclid's geometry hardly discussed by Hartshorne (Hartshorne 2000, 166–167). Here is a brief account (Błaszczyk, Mrówka 2013, ch. 3).

Euclidean proportion (for which we adopt symbol :: originated from the 17th-century) is a relation between two pairs of geometric figures (*megethos*) of the same kind, triangles being of one kind, line segments of another kind, angles of yet another. Magnitudes of the same kind form an ordered additive semi-group $\mathfrak{M} = (M, +, <)$ characterized by the five axioms given below (Błaszczyk, Petiurenko 2019, § 3.1.).

E1 $(\forall a, b \in M)(\exists n \in \mathbb{N})(na > b)$.

E2 $(\forall a, b \in M)(\exists c \in M)(a > b \Rightarrow a = b + c)$.

E3 $(\forall a, b, c \in M)(a > b \Rightarrow a + c > b + c)$.

E4 $(\forall a \in M)(\forall n \in \mathbb{N})(\exists b \in M)(nb = a)$.

E5 $(\forall a, b, c \in M)(\exists d \in M)(a : b :: c : d)$, where $na = \underbrace{a + a + ... + a}_{n-times}$.

Clearly, E1–E3 provide extra characteristics of the *greater-than* relation; while E1 is a sheer rendition of definition V.4, currently called the Archimedean axiom.

A modern interpretation of *Common Notions* is simple: CN 1 justifies the transitivity of congruence of line segments, triangles, and angles, CN 2 and 3 – addition and subtraction in the following form

$$a = a',\ b = b' \Rightarrow a + b = a' + b', \quad a - a' = b - b'.$$

The famous CN 5, *Whole is greater than the part*, allows an interpretation by the formula

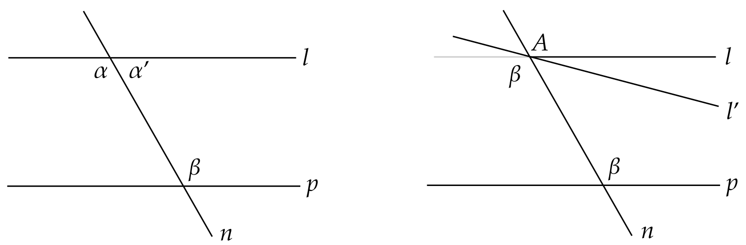


Figure 1.31: *Elements*, I.29 schematized (left). Hartshorne's version (right)

$a + b > a$ (Błaszczyk, Mrówka, Petiurenko 2020, 73–76).[7]

In the Hilbert system, the *greater-than* relation is defined through the concept of *betweeness* and refers only to line segments and angles (Hartshorne 2000, 85, 95); similarly, addition of line segments and angles is introduced by definitions (Hartshorne 2000, 168,93). Then counterparts of Euclid's axioms E2, E3, CN 1–3 are proved as theorems.

In the sequel, we still will be referring to our interpretation of Euclid's *greater-than* relation, therefore already at that stage, juxtapose Euclid's and Harthstone's proof of I.29 as a model clash of these alternative approaches. Its substance is as follows: *When a line n falls across parallel lines l, p, equality of angles obtains* $\alpha = \beta$ (see Fig. 1.31, left). And Euclid's proof goes like that: For, if they are not equal, one of the angles is greater, suppose $\alpha > \beta$. Then (implicitly by E3),

$$\alpha > \beta \Rightarrow \alpha + \alpha' > \beta + \alpha',$$

given that α, α' are supplementary angles.

Since $\alpha + \alpha' = \pi$, angles β, α' satisfy the requirement of the parallel axiom, i.e., $\beta + \alpha' < \pi$ and straight lines l, p meet, contrary to the initial assumption.

On the other hand, Hartshorne's proof of I.29 rests on the parallel axiom stating there is exactly one line through the point A parallel to p (see Fig. 1.31, right). Then, if $\alpha \neq \beta$, he constructs a line l' through A making angle β with n, which, by I.27, is parallel to p – it

[7] Avigad et al., (Avigad, Dean, Mumma 2009, 722), also adopt that interpretation of CN5, but out of nothing, i.e., without any reference to the *Elements*. On another occasion, they interpret CN5 as an inclusion of areas (Avigad, Dean, Mumma 2009, 704). Indeed, it is controversial to imply that by checking a diagram, one could confirm that $a + b > a$. Yet, on p. 744, they suggest that one can read off a diagram that part makes sum $a + b$, and based on that observation – we guess – one can infer that $a + b > a$.

contradicts the uniqueness of a parallel line through A.

Euclid's proof, thus, implies an intersection point of lines l and p, Hartshorne's – a second parallel line to p.

1.2.4 Perpendicular lines. I 9–12

Two subsequent propositions provide bisection of an angle and a line segment. Then Euclid constructs a perpendicular to a line through a point lying on it and outside it. The SSS rule is applied to justify these constructions.

I.9 *To cut a given rectilinear angle in half.*

Taking D on AB, point E is such that $AD = b = AE$ (see Fig. 1.32). Point F is determined by I.1, taking $DE = a$. Then by SSS, $\triangle FDA = \triangle FEA$. Hence, $2\beta = \alpha$.

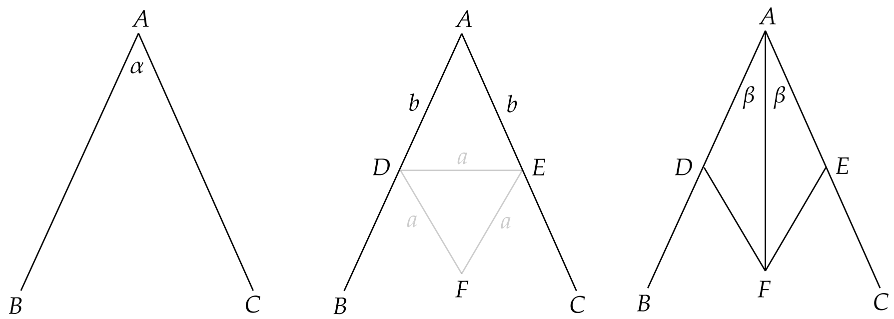


Figure 1.32: Proof of I.9 schematized

$AB^{\rightarrow}$	$(A, b),\ AC^{\rightarrow}$	$(D, a),\ (E, a)$
D	E	F

Euclid's diagram implies $a < b$. If $a > b$, the line AF also divides α in half. When $a = b$, the construction does not produce a new point, given that I.1 produces only one point. I.8 implies the construction of congruent triangles on both *sides* of DE.

I.10. *To cut a given finite straight-line in half.*

On both sides of AB, construct equilateral triangles $\triangle ABC$ and $\triangle ABF$ (see Fig. 1.33). By I.9, CF divides in half angle $\angle ACB$. Then, by SAS, $\triangle ACD = \triangle BCD$, hence $AD = DB$.

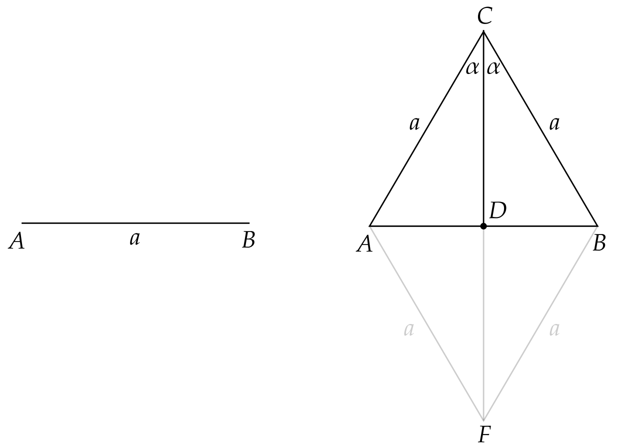


Figure 1.33: Proof of I.10 schematized

$(A,a),\ (B,a)$	$CF,\ AB$
$C,\ F$	D

Note that point D occurs as an intersection of two straight lines. There is no explicit rule in the Euclid system to guarantee its existence. In the Hilbert system, it follows from the so-called cross-bar theorem, a simple follow-up of the Pasch axiom (Hartshorne 2000, 77–78).[8]

I.11 *To draw a straight-line at right-angles to a given straight-line from a given point on it.*

Taking a random point D on the half-line $CA^{\rightarrow}$, E is determined such that $CD = CE$, and F is the vertex of equilateral triangle DFE (see Fig. 1.34). By I.8, $\triangle DFC = \triangle EFC$. Hence, $\angle DCF = \angle ECF$. Since they are equal and supplementary angles, by I def. 10, they are both right angles.

$CA^{\rightarrow}$	$(C,a),\ AB$	$(D,2a),\ (E,2a)$
D	E	F

I.12 *To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.*

D is a random point "on the other side [to C] of the straight-line AB" (see Fig. 1.35). By

[8]One can also justify the existence of point D owning to the concept of side of straight line (see § 3.8 below): the points C and F are known to lie on different sides of the line AB, and thus, the segment CF intersects the line AB.

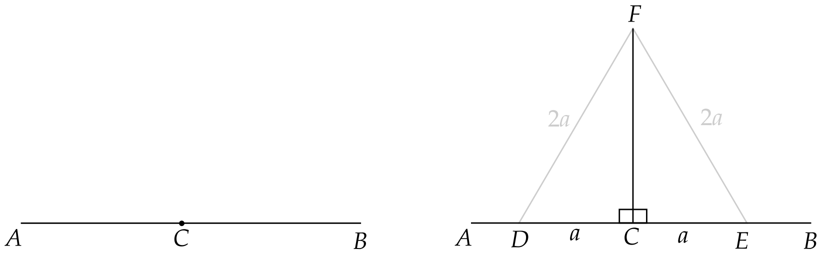


Figure 1.34: Proof of I.11 schematized

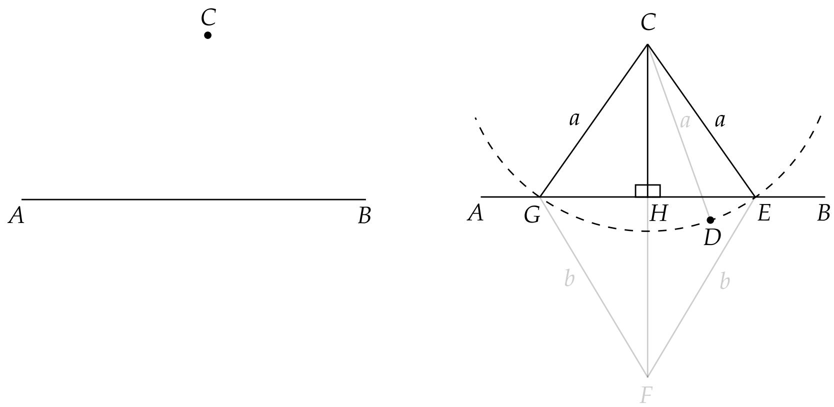


Figure 1.35: Proof of I.12 schematized

I.10, H is determined such that $GH = HE$. By SSS, $\triangle GCH = \triangle ECH$ and by the same argument as in I.11, $\angle CHG = CHE = \pi/2$.

	$AB, (C, a)$	$(G, b),\ (E, b)$	$CF,\ AB$	
AB, C	D	$G,\ E$	F	H

Euclid does not definite *the side of line* – concept involved in I.8 and I.12. Modern axiomatics reveal the importance of that concept.

1.2.5 A comment on Postulate 2

In the Hilbert system, due to B2, the straight line has no ends. In most of Euclid's propositions, a straight line is a line segment with endpoints – a closed line segment, by modern standards. That is why the term *line standing on another line* involved in propositions I.13–14 makes sense;[9] in Fig. 1.37, lines EB and AB stand on DC. Such a line can stand at point B that is between D and C (see Fig. 1.37), or at the endpoint, such as AB stands on CB in Fig. 1.38. Now, line *standing on* another line enables Euclid to formulate iff-condition for a line to be an extension of another line, and that is a job of propositions I.13–14, while Postulate 2 characterizes the process of extending in an informal, rather a descriptive way, namely *To produce a finite straight-line continuously in a straight-line.*

Similarly, in propositions I.27–29, Euclid introduces an auxiliary line enabling him to eliminate a clumsy condition *being produced to infinity* included in definition of parallel lines (I def. 23). Given that symbol $l \dot{-} p$ stands for l *is straight on with respect to* p, we can represent I.27–29 and I.14 as follows (see Fig. 1.36)

$$l \parallel p \Leftrightarrow \alpha + \beta = \pi, \quad l \dot{-} p \Leftrightarrow \alpha + \beta = \pi.$$

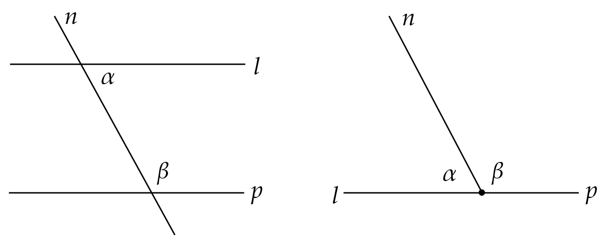


Figure 1.36: Iff-conditions for parallelism and extension of a line

The symbol $AB^{\rightarrow}$, we have already applied in tables, evoking modern half-line, actually stands for a extension of the line segment AB.

1.2.6 Vertex angles. I 13–15

I.13 *If a straight-line stood on a straight-line makes angles, it will certainly either make two right-angles, or equal to two right-angles.*

If $\angle CBA = \angle ABD$ (see Fig. 1.37), by I def.10, they are two right angles. If $\angle CBA \neq \angle ABD$, Euclid tacitly assumes $\angle ABD > \pi/2$, draws a perpendicular $EB \perp DC$ and argues: since the following equalities of angles obtain

$$(\beta + \gamma) + \alpha = \gamma + \beta + \alpha,$$

[9]The very concept is also employed in the definition of the right angle, I def. 10.

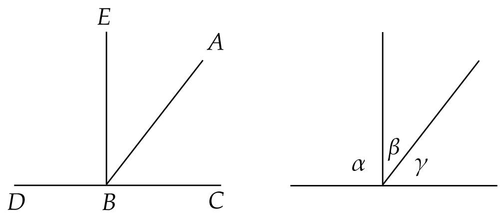


Figure 1.37: *Elements*, I.13 (left)

$$(\alpha + \beta) + \gamma = \alpha + \beta + \gamma,$$

then, by CN1,

$$(\beta + \gamma) + \alpha = (\alpha + \beta) + \gamma.$$

Hence, $\angle ABD + \angle CBA = (\alpha + \beta) + \gamma = \pi$.[10] □

Instead of Euclid's embroiled enunciation of proposition I.14, we offer the following paraphrase: If $\alpha + \beta = \pi$, then l is straight on to p, given that n stands on l at its end (see Fig. 1.38, middle).

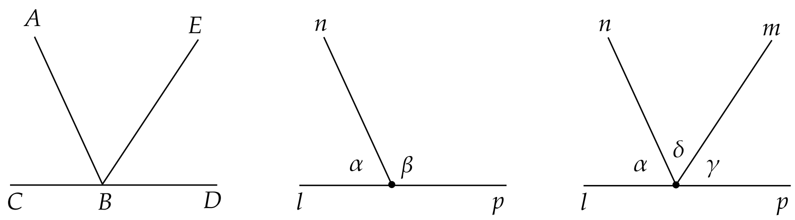


Figure 1.38: *Elements*, I.14 (left), thesis (middle), and its schematized proof (right)

The proof adopts *reductio ad absurdum* mode. For if not, let m be straight on to l; Postulate 2 guarantees the existence of m (see Fig. 1.38, right). Then, by I.13, $\alpha + \delta = \pi$. Since $\delta + \gamma = \beta = \angle(n, p)$, the following equality holds $\alpha + \delta + \gamma = \pi$. Hence

$$\alpha + \delta = \alpha + \delta + \gamma.$$

By CN3, $\delta = \delta + \gamma$, and Euclid continues: *the lesser to the greater. The very thing is impossible.* □

[10] However, there is no demonstration whatsoever that $\gamma + \beta + \alpha = \alpha + \beta + \gamma$.

With our interpretation of CN5, $\delta + \gamma > \delta$, and due to the trichotomy law, it can not be both $\delta = \delta + \gamma$ and $\delta + \gamma > \delta$.

Euclid applies criterion I.14 in I.15 and also in I.47.

I.15 *If two straight-lines cut one another then they make the vertically opposite angles equal to one another.*

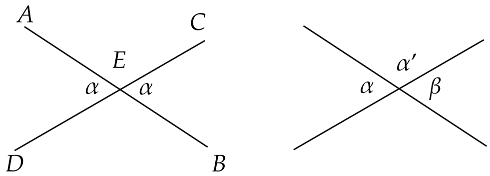


Figure 1.39: Proof of I.15

By I.13 (see Fig. 1.39),

$$\alpha + \alpha' = \pi = \alpha' + \beta.$$

By CN3, $\alpha = \beta$. $\square$

The condition $\alpha + \alpha' = \pi$ follows from the assumption AE *stands on* DC, while $\alpha' + \beta = \pi$ – from the assumption CE *stands on* AB. To be more Euclidean, α and α' make two right angles, similarly α' and β make two right angles. By Postulate 4, *all right-angles are equal to one another.* Hence $\alpha + \alpha' = \alpha' + \beta$, and the conclusion follows.

1.2.7 Side of straight-line

In (Hartshorne 2000, § 7), Hartshorne introduces the concept *side of line* l. It is an equivalence relation between points of plane not lying on l defined by: $A \sim B$ iff $A = B$ or segment AB does not meet l. It determines two equivalence class, called sides of l, or half-planes.

The transitivity of the relation is demonstrated as follows (see Fig. 1.40, left). Let $A \sim C$, and $B \sim C$ and suppose $A \nsim B$. Let $D = l \cap AB$. Then, by Pasch axiom, l intersects AC or BC, which contradicts $A \sim C$ or $B \sim C$, respectively.

Euclid's straight-line is not that *long* and can not divide the plane into two halves. Postulate 2 guarantees that one can extend l to, say, point E, but neither Postulate 1 nor 3 can guarantee an intersection l with AC or BC (see Fig. 1.40, right).

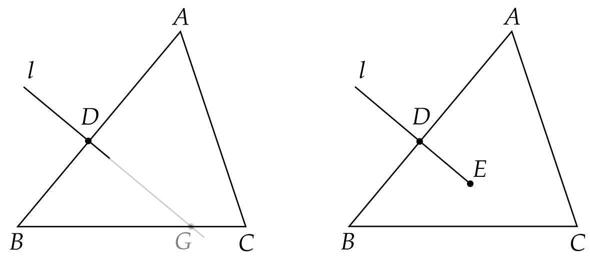


Figure 1.40: Transitivity of the relation the *same side of line* (left), Euclidean version (right)

Since there are three noncollinear points, relation $\sim$ determines at least one equivalence class. Hartshorne shows there are at most two classes. Let us consider a simpler question on whether there are two different classes. Let $A \notin l$, $E \in l$, then, by B2, there is F such that $A * E * F$ (see Fig. 1.41, left). Hence, A and F lie on different sides of l.

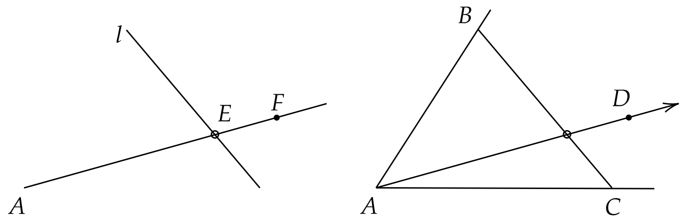


Figure 1.41: Finding points on different sides of l (left) and crossbar theorem (right)

Crossbar theorem (Hartshorne 2000, 77–78), Fig. 1.41, right), that builds on the Pasch axiom and the concept of side of the straight line, enables to infer the existence of points such as D or H in Euclid's propositions I.10, 12. Furthermore, the concept of side of straight-line is crucial in proposition I.12, as "D have been taken somewhere on the other side (to C) of the straight-line AB". Indeed, if D is taken at random on the same side with C, Euclid's construction would not work, for D could be on the perpendicular CH (see Fig. 1.35).

In (Hartshorne 2000), the concept of supplementary angles and the five-segment-lines theorem cover Euclid's propositions I.13 and I.15 (Hartshorne 2000, 92–93). As for I.14, Hartshorne finds it completely alien to Hilbert's concepts and proposes an exercise "to rewrite the statement I.14 so that it makes sense in the Hilbert plane" (Hartshorne 2000, 103). (Beeson, Narboux, Wiedijk 2019, 218), finds it simply as a statement on the betweenness relation; (Mueller 2006, 20), – a *converse* of I.13.

Let us review I.14 in the context of propositions I.13–15 and find what makes cutting lines look like in I.15 rather than in the proof of I.14. To this end, let m be a continuation of n, and p of l. Both n and m stand on $l\dot{-}p$, also l and p *stand on* $n\dot{-}m$ (Fig. 1.42). Furthermore, the angles between l and n, and n and p add up to π;

$$\angle(l,n) + \angle(n,p) = \pi, \quad \angle(l,m) + \angle(m,p) = \pi.$$

These results fit to I.14. However, taking into account that p also *stands on* the line $n\dot{-}m$, one obtains

$$\angle(m,p) + \angle(p,n) > \pi,$$

since $\angle(p,n) > \angle(p,l) = \pi$. It contradicts I.14.

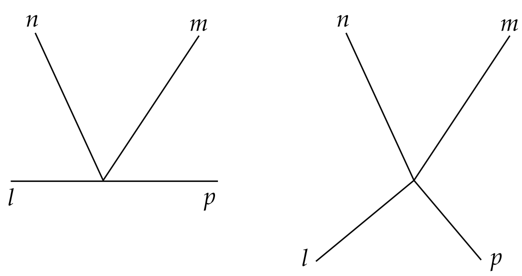


Figure 1.42: Weird straight-lines

1.2.8 Triangle inequality. I 16-21

That set of propositions, with proofs on a regular basis referring to transitivity, trichotomy law, axioms E2, E3, and Common Notions 5, is a festival of *greater-than* relation.

I.16 *For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.*

	$BC^{\rightarrow}$	mid AC	$BE^{\rightarrow}$, (E,b)	$AC^{\rightarrow}$
$\triangle ABC$	D	E	F	G

Point E is the middle of AC, $AE = a = EC$; F is such that $BE = b = EF$. By I.15, $\angle AEB = \angle FEC$. Hence, by SAS, $\triangle AEB = \triangle FEC$ (See Fig. 1.43, triangles in grey), and

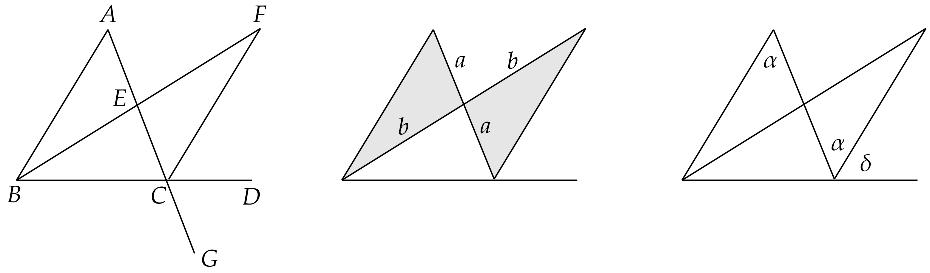


Figure 1.43: *Elements*, I.16 (left) and its schematized proof (middle and right)

angles at vertexes A and C are equal, $\angle A = \alpha = \angle C$. Now,

$$\alpha + \delta > \alpha,$$

meaning the exterior angle $\angle ACD$ is greater than the interior angle $\angle BAE$.

The same argument applies to angles $\angle ABC$ and $\angle BCG$, but $\angle BCG = \angle ACD$, thus the thesis obtains. □

I.17 *For any triangle, two angles are less than two right-angles.*

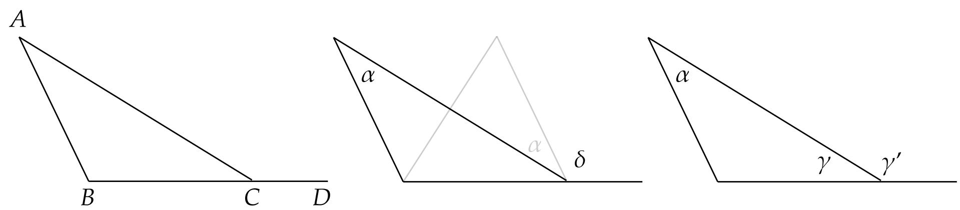


Figure 1.44: *Elements*, I.17 (felt), shadow construction I.16 (middle), schematized proof (right)

By I.16, $\alpha < \gamma'$ (see Fig. 1.44, right). Adding to both sides γ, we obtain

$$\alpha + \gamma < \gamma' + \gamma.$$

Since $\gamma' + \gamma = \pi$, the required inequality holds, $\alpha + \gamma < \pi$. □

I.18 *For any triangle, the greater side subtends the greater angle.*

In symbols (see Fig. 1.45, middle)

$$c > a \Rightarrow \gamma > \alpha.$$

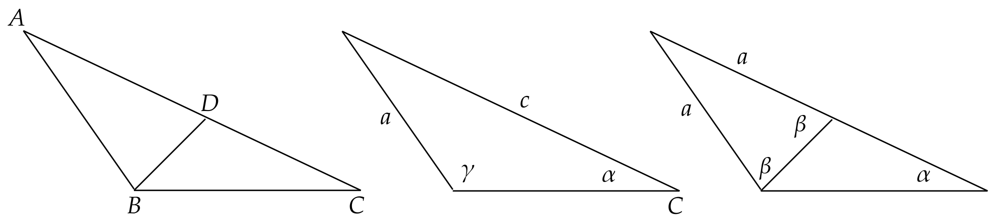


Figure 1.45: *Elements*, I.18 and its schematized proof

$$\begin{array}{r|c} & AC, (A, a) \\ \hline \triangle ABC & D \end{array}$$

If $AC > AB$, there is point D such that $AD = a = AB$. In triangle $\triangle ABD$, angles at the base are equal (see Fig. 1.45, right). By I.16, $\beta > \alpha$. By transitivity

$$\beta > \alpha, \ \gamma > \beta \Rightarrow \gamma > \alpha.$$

Inequality $\beta > \alpha$ is determined at point D; inequality $\gamma > \beta$ – at the vertex B. □

While $c > a \Rightarrow \gamma > \alpha$ represents I.18, the reverse implication $\gamma > \alpha \Rightarrow c > a$, represents I.19 (see Fig. 1.46). Hence, I.18-19 bring in the equivalence

$$c > a \Leftrightarrow \gamma > \alpha.$$

I.19 *For any triangle, the greater angle is subtended by the greater side.*

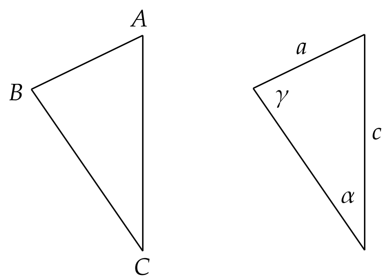


Figure 1.46: *Elements*, I.19 (left)

The proof builds on the trichotomy law. If c is not greater than a, then $c = a$ or $c < a$. From the first case, by I.5, equality follows $\gamma = \alpha$. From the second, by the previous proposition, $\gamma < \alpha$. Both cases contradict the supposition $\gamma > \alpha$. □

I.20 *For any triangle, two sides are greater than the remaining (side).*

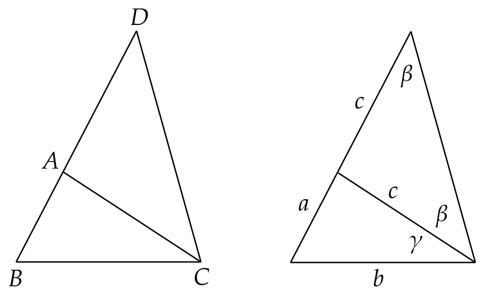


Figure 1.47: *Elements*, I.20 and its schematized proof

	$AB^{\rightarrow}$, (A, c)
$\triangle ABC$	D

Point D is constructed on $BA^{\rightarrow}$ such that $AD = c = AC$. Hence in triangle $\triangle ADC$, angles at the base are equal (see Fig 1.47, right). Since $\gamma + \beta > \beta$, in triangle BDC, by I.19, $a + c > b$. With regard to other pairs of sides one proceeds similarly. □

I.21 *If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle.* In symbols:

$BA + AC > BD + DC$ and $\angle A < \angle D$ (see Fig. 1.48, left).

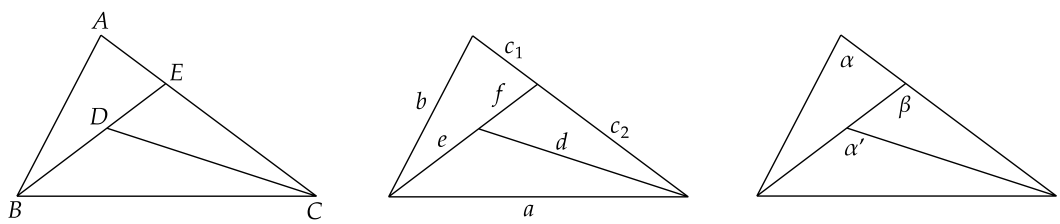


Figure 1.48: *Elements*, I.21 and its schematized proof

The proof is an exercise in the already proved triangle inequality. Given that $c + c_1 + c_2$, we have

$$e + f < b + c_1 \Rightarrow e + f < b + c_1 + c_2,$$

$$d < f + c_2 \Rightarrow e + d < e + f + c_2.$$

Hence

$$e + d < b + c.$$

For the second part, Euclid applies twice I.16 and transitivity as follows

$$\alpha' > \beta,\ \beta > \alpha \Rightarrow \alpha' > \alpha.$$

□

1.2.9 Transportation of angles. I 22–23

In I.22, Euclid builds a triangle from three given line segments.[11]

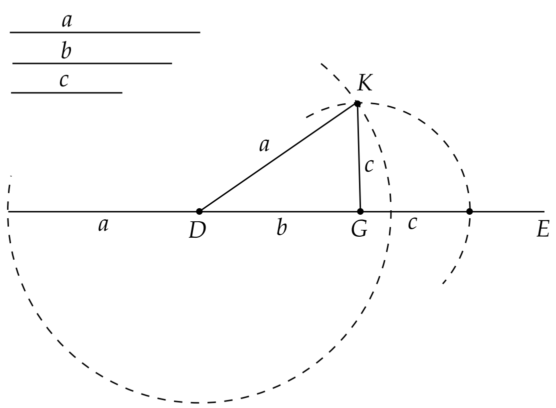


Figure 1.49: *Elements*, I.22 – small letters added

The below table presents G, K as intersection points. Let us remind that symbol Da stands for transportation of line segment a to point D based on I.2.

	Db	(D,b), DE	Da	Gc	(D,a), (G,c)
DE		G			K

The reminder of the proof includes justifications of equalities $FK = a$, $FG = b$, $GK = c$. □

[11] (Greenberg 2008, 173), observes it is equivalent to the circle-circle axiom.

I.23 *To construct a rectilinear angle equal to a given rectilinear angle at a point on a given straight-line.*

The transportation of an angle α is reduced to a transportation of triangle $\triangle DCE$ (see Fig. 1.50).

Figure 1.50: *Elements*, I.23 – small letters added

The construction part consists of picking two random points on arms of the angle.

$CK^{\rightarrow}$	$CE^{\rightarrow}$
D	E

Then triangle $\triangle DCE$ is copied at point A on the line AB (see Fig. 1.51).

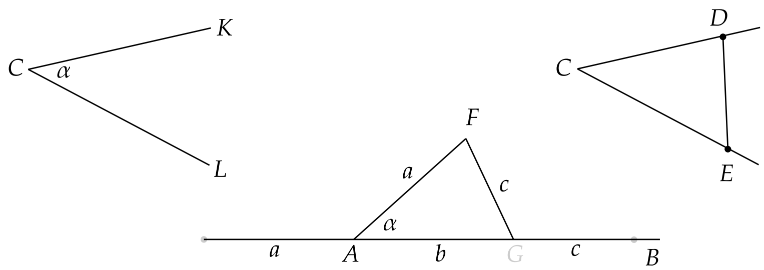


Figure 1.51: Copying an angle

By the SSS, $\triangle CDE = \triangle AFG$, hence $\angle KCL = \alpha = \angle FAG$. $\square$

1.2.10 ASA and SAA rules. I 24–26

Propositions I.24–25 are companions to I.18–19, yet, this time, Euclid considers two separate triangles. We present their thesis in concise, symbolic forms.

I.24 Let $AC = a = DF$, $AB = b = DF$, and $\angle CAB = \alpha$, $\angle FDE = \beta$. If $\angle CAB > \angle FDE$, then $CB > EF$ (see Fig. 1.52). In small letters mode

$$\alpha > \beta \Rightarrow c > d.$$

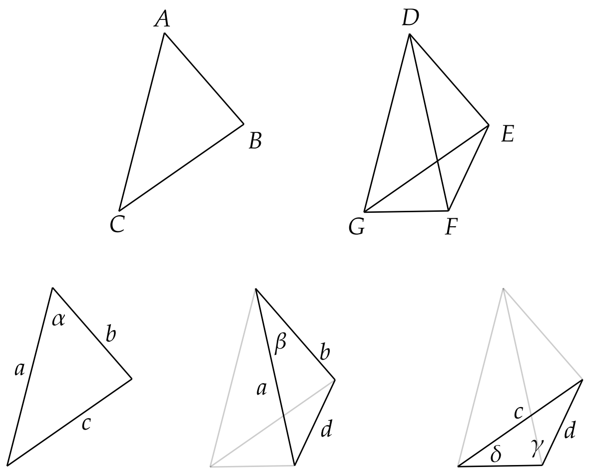


Figure 1.52: *Elements*, I.24 and its schematized proof

Since triangle $\triangle DGF$ is isosceles, the equality of angle holds $\angle DGF = \angle DFG$. Hence, in triangle $\triangle GFE$, $\delta < \gamma$. By I.18, $d < c$, meaning $FE < CB$. □

I.25 Let $AB = a = DE$, $AC = b = DF$, and $\angle BAC = \alpha$, $\angle EDF = \beta$. If $CB > EF$, then $\angle CAB > \angle FDE$ (see Fig. 1.53). In small letters mode

$$c > d \Rightarrow \alpha > \beta.$$

The proof is an exercise in trichotomy law and goes like that. If it is not that $\alpha > \beta$, then $\alpha = \beta$ or $\alpha < \beta$. In the first case, $c = d$. In the second, by I.24, $c < d$. □

I.26 *If two triangles have two angles equal to two angles, respectively, and one side equal to one side then (the triangles) will also have the remaining sides equal to the remaining sides, and the remaining angle (equal) to the remaining angle.*

It is ASA congruence rule: If $BC = EF$, $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, then $\triangle ABC = \triangle DEF$ (see Fig. 1.54).

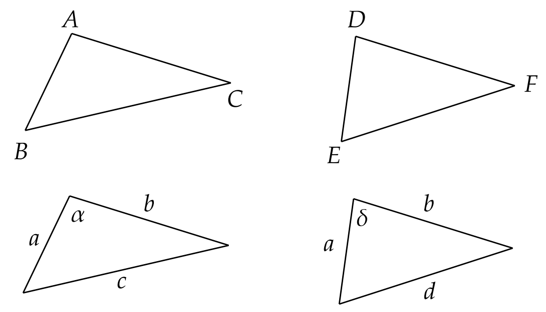


Figure 1.53: *Elements*, I.25

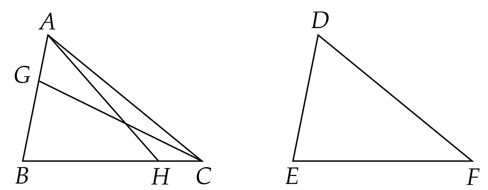


Figure 1.54: *Elements*, I.26

Let $AB = b$, $DE = b'$ (see Fig. 1.55). Supposing $b' < b$, Euclid lays down b' on AB, and by SAS rule, gets the equality of triangles $\triangle GBC = \triangle DEF$. Hence, the equality of angles follows $\angle GCB = \angle ABC$, *the lesser to the greater.*

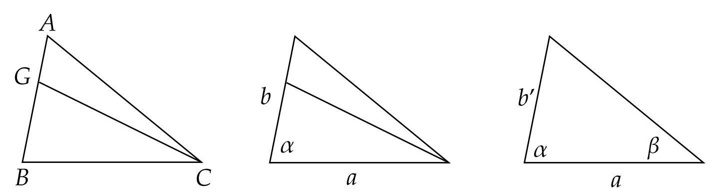


Figure 1.55: *Elements*, I.26 - scheme of the first case proof

In the second case, SAA, $AB = DE$, $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$. Suppose $BC > EF$. Let $BN = b' = EF$ (see Fig. 1.56). Thus, $\triangle AHB = \triangle DFE$, and, on the one hand $\angle AHB = \angle DFE$, on the other, by I.16, $\angle AHB > \angle DFE$, *the very that is impossible.*

$\square$

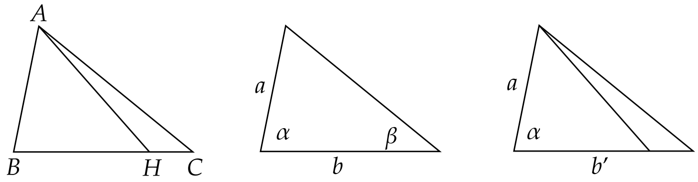


Figure 1.56: *Elements*, I.26 - scheme of the second case proof

1.2.11 Parallel lines. I 27–31

Until proposition I.29, Euclid's arguments do not rely on the parallel postulate, yet, in I.27, aiming to show $AB \parallel CD$, given that $\angle AEF = \angle EFD$ (see Fig. 1.58, left) he invokes definition of parallel lines: "Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither" (I def. 23).

I.27 *If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the straight-lines will be parallel to one another.* In symbols, $\alpha = \beta \Rightarrow p \parallel l$ (see Fig. 1.57).

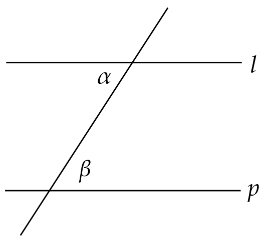


Figure 1.57: Simplified version of I.27

The proof proceeds in *reductio ad absurdum* mode and starts with the claim: "if not, being produced, AB and CD will certainly meet together". Suppose, thus, AB and CD are not parallel and meet in G (see Fig. 1.58, right). Then, in triangle EFG, the external angle $\angle AEF$ is equal to the internal and opposite angle $\angle EFD$, but, by I.16, $\angle AEF$ is also greater than $\angle EFD$. Hence, $\angle AEF = \angle EFD$ and $\angle AEF > \angle EFD$. *The very thing is impossible.*

□

The rationale for point G lies in the definition of parallel lines rather than in construction with a straightedge and compass. Thus, next to I.7, it is another non-constructive proposition

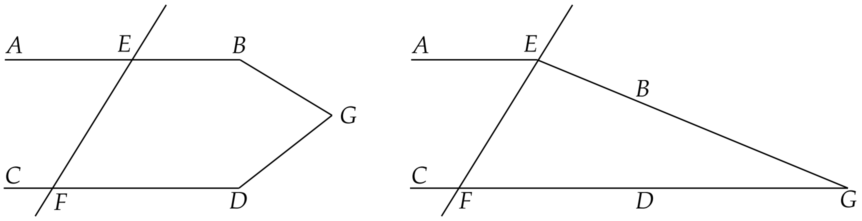


Figure 1.58: *Elements*, I.27 (left) and a triangle implied in its proof (right)

of the *Elements*.

I.28 *If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, then the (two) straight-lines will be parallel to one another.* In symbols, $\alpha = \beta \Rightarrow l \parallel p$ (see Fig. 1.59).

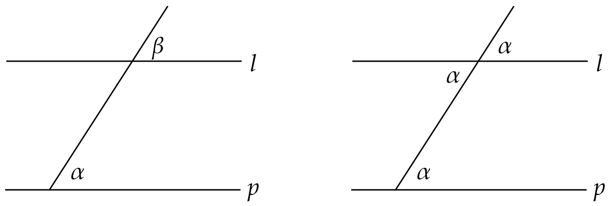


Figure 1.59: Simplified version of I.28 (left) and its proof (right)

The proof refers to I.27 and the equality of vertical angles (see Fig. 1.59, right).

I.29 *A straight-line falling across parallel straight-lines makes the alternate angles equal to one another.* In symbols (see Fig. 1.60, right)

$$p \parallel l \Rightarrow \alpha = \beta.$$

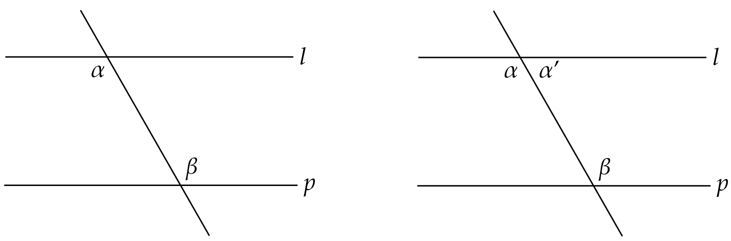


Figure 1.60: Simplified version of I.29 (left) and its proof (right)

To get a contradiction, suppose $\alpha \neq \beta$. Hence, one of the angles is greater. Let $\alpha > \beta$. Then (implicitly by E3),

$$\alpha > \beta \Rightarrow \alpha + \alpha' > \beta + \alpha'.$$

Since $\alpha + \alpha' = \pi$, angles β, α' satisfy the requirement of the parallel axiom, i.e., $\beta + \alpha' < \pi$ and straight lines l, p meet, contrary to initial assumption.

□

In I.29, Euclid applies Postulate 5 for the first time. It reads:

And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles.[12]

In the Fig. 1.60, angles β, α' satisfy – let us repeat – the requirement of that Postulate, that is $\beta + \alpha' < \pi$. In definition I.23, parallel lines on a plane are characterized by the condition *being produced to infinity do not meet*; Postulate 5 includes such a condition that when it is satisfied, makes lines intersect when *being produced to infinity.*

Due to propositions I.27–29 we get the following equivalence (see Fig. 1.61)

$$l \parallel p \Leftrightarrow \alpha = \beta.$$

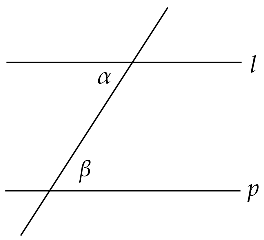


Figure 1.61: Characteristic of parallel lines

From I.29 on, Euclid applies the above characteristics of the parallel lines and does not rely on the proviso *being produced to infinity* any more.

I.30 *(Straight-lines) parallel to the same straight-line are also parallel to one another.*

[12](De Risi 2016) enlists versions of the Postulate 5 throughout early modern and modern editions.

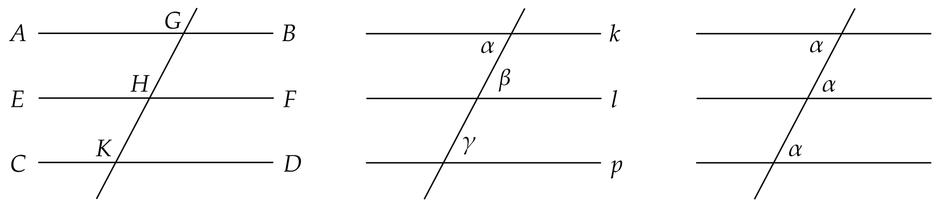


Figure 1.62: *Elements*, I.30 (left) and its schematized proof (middle and right)

Since $k \parallel l$, by I.29, $\alpha = \beta$ (see Fig. 1.62, middle). Similarity, since $l \parallel p$, by I.29, $\beta = \gamma$. By CN1, $\alpha = \gamma$. Hence, finally, by I.27, $k \parallel p$. □

The proof seems simple, built on I.27–29 and the transitivity of equality. However, with no discussion, Euclid assumes the existence of the line GK falling on the three parallel lines (see Fig. 1.62, left).[13]

I.31 *To draw a straight-line parallel to a given straight-line, through a given point.*

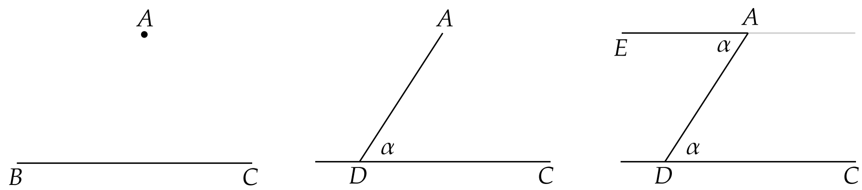


Figure 1.63: Proof of I.31 schematized

It is a sheer construction-type proposition. On the straight-line BC, Euclid picks a random point D and copies the angle $ADC = \alpha$ at point A. By I.27, line EA is parallel to DC (see Fig. 1.63 and the below table).

$BC^{\rightarrow}$	$AD\alpha$
D	E

In sum, proposition I.27 provides grounds for the existence of a line parallel to p through a point A not lying on p; therein, 'parallel' means not intersecting p. Due to I.29, it is the only line through A not meeting p. These two propositions justify Hilbert's version of Euclid's

[13] That assumption, in the case in which all three lines are pairwise parallel, is equivalent to the *Lotschnittaxiom*, as shown in (Pambuccian, Schacht 2021), Theorem 5.2.

axiom: There is at most one line parallel to p through A. Since I.27 holds in the absolute geometry, we can also use a more efficient version, namely: There is exactly one line parallel to p through A.

1.2.12 Sum of angles in triangle. I 32

I.32 *Three internal angles of the triangle, ABC, BCA, and CAB, are equal to two right-angles.*

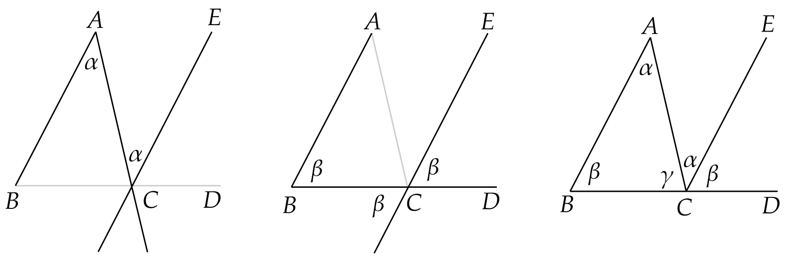


Figure 1.64: Proof of I.32 schematized

To proof the thesis, Euclid transports angle α to point C, and draws CE, which, by I.27, is parallel to AB (see Fig. 1.64). Hence, by I.29, $\angle ECD = \beta$, and angles at C sum up to "two right angles", $\beta + \alpha + \gamma = \pi$.

The construction part comprises to drawing parallel line to AB through the point C.

$$\frac{AB||C}{E}$$

Note that, concerning the angle α, AC is an auxiliary line falling on AB and EC, regarding β – BD is such a line. Thus, line EC is parallel to the side AB, while other sides of the triangle play the role of auxiliary lines. It seems possible that in that proof lies Euclid's idea of *Postulate* V.

1.2.13 Parallelograms. I 33–34

I.33 *Straight-lines joining equal and parallel on the same sides are themselves also equal and parallel.*

In that and the subsequent proposition, straight-line stands for a line segments, thus rendered in symbols, I.33 reads (see Fig. 1.65, middle and right),

$$l \parallel p,\ l = p\ \Rightarrow\ q \parallel s,\ q = s.$$

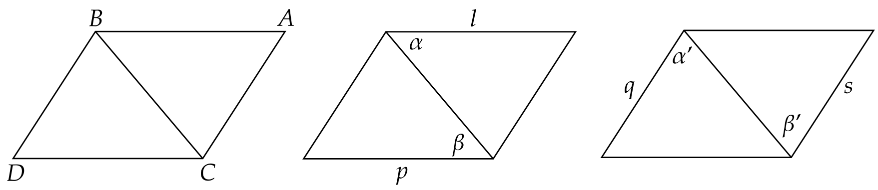


Figure 1.65: *Elements* I.33 and its schematized proof (middle and right)

From the assumption $l \parallel p$, by I.29, it follows that $\alpha = \beta$. Due to SAS, the equality of triangles obtains $\triangle ABC = \triangle DCB$. Hence, $\alpha' = \beta'$ and $q = s$. Finally, since $\alpha' = \beta'$, by I.27, $q \parallel s$. $\square$

I.34 *For parallelogrammic figures, the opposite sides and angles are equal to one another and a diagonal cuts them in half.* In symbols (see Fig. 1.66, right),

$$l \parallel p,\ q \parallel s \Rightarrow\ l = p,\ q = s.$$

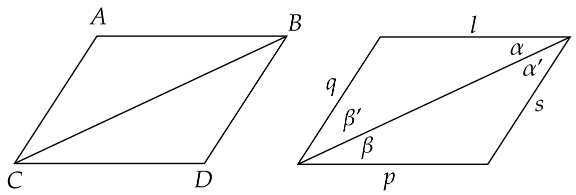


Figure 1.66: *Elements* I.34 (left) and its schematized proof (right)

Since $l \parallel p$, by I.29, $\alpha = \beta$. Similarly, from $q \parallel s$, the equality of angles follows $\alpha' = \beta'$. Due to ASA rule, the equality of triangles $\triangle ABC = \triangle DCB$ obtains. As a result $l = p$ and $q = s$. $\square$

1.2.14 Theory of equal figures. I 35–45

Euclid's theory of equal figures (theory of area) is a set of propositions enabling the transformation of a (convex) polygon A into a square S, meeting the requirement $A = S$. While congruence of figures is based on *Common Notions* 4, the equality of non-congruent figures is a procedural one: $A = B$ iff there is a series of figures $A_1, ..., A_n$ such that $A_1 = A$, $A_i = A_{i+1}$, $A_n = B$, while equalities $A_i = A_{i+1}$ are guaranteed by *Common Notions* and *Postulates* 1 to 3.[14] Since (Błaszczyk 2018) details the theory, below we only sketch it.

The starting point of the theory, proposition I.35, states the equality of parallelograms $ADCB$ and $EFCB$ *which are on the same base and between the same parallels*, i.e., with the same height (see Fig. 1.67, upper left). Its proof proceeds as follows: By I.34 and SSS, triangles AEB, DFC are equal. When triangle T_1 is subtracted from each of them, the remainders $ADGB$ and $EFCG$ are equal by CN3, although non-congruent. When triangle T_2 is added to both $ADGB$ and $EFCG$, *the whole parallelogram* $ABCD$ *is equal to the whole parallelogram* $EFCB$, by CN2.

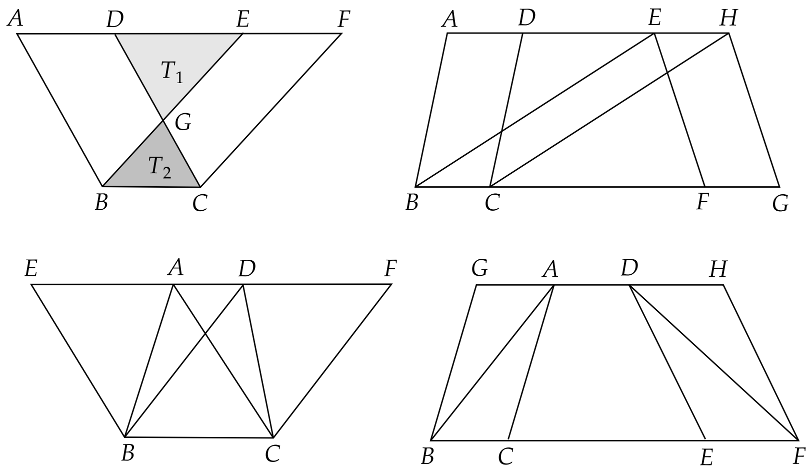


Figure 1.67: *Elements*, I.35–38

In proposition I.36 (Fig. 1.67, upper right), Euclid shows the same result for parallelograms on equal bases BC and FG. His argument relies on the transitivity of equality guaranteed by CN1 and equalities based on I.35, namely, $ADBC = EHCB$, and $EHCB = EHFG$.

Propositions I.37–38 (Fig. 1.67, bottom row) reiterates the same results regarding triangles *on the same base* and *on equal bases* respectively. In both cases, Euclid considers triangles as halves of respective parallelograms.

In I.41, Euclid demonstrates that a parallelogram on the same base and between the same

[14]The exhaustion method, developed in Book XII, brings in yet another meaning of equal figures.

parallels as a triangle is the *double of the triangle*. Proposition I.42 provides a construction of a parallelogram equal to the given triangle ACB. Its proof consists of finding the midpoint E on the base BC and drawing parallel lines through E and C (see Fig. 1.68, right). FEC in the resulting parallelogram $FGCE$, could be any angle; therefore, a triangle can be transformed into, e.g., a rectangle. The height of the parallelogram obtained through that construction and that of the triangle are equal – both figures *are between the same parallels*.

In I.44, Euclid finds a parallelogram with a fixed height equal to a given parallelogram. As a result, a triangle can be transformed into an equal rectangle, while heights of these figures differ.

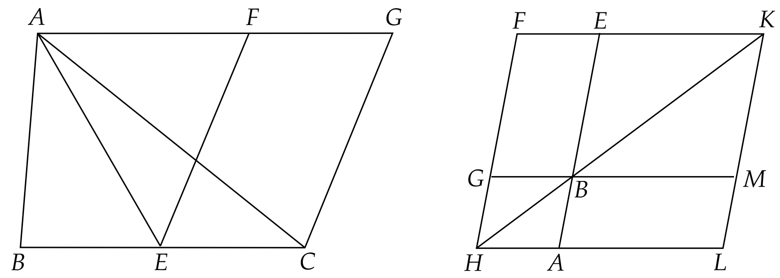


Figure 1.68: *Elements*, I.42, 44

Let the parallelogram $FEGB$ be given (Fig. 1.68, right). The construction runs as presented in the below table, where A is taken on the half-line $EB^{\rightarrow}$ at will, meaning one can adjust the height of the parallelogram $BMLA$ at will.

$EB^{\rightarrow}$	$FG^{\rightarrow}$, $A \parallel GB$	$FE^{\rightarrow}$, $HB^{\rightarrow}$	$K \parallel FG$, $GB^{\rightarrow}$	$HA^{\rightarrow}$, $KM^{\rightarrow}$
A	H	K	M	L

The equality of parallelograms $FEGB = BMAL$ follows from a subtraction from equal triangles $\triangle FKH = \triangle LKH$, at first, equal triangles EBK and BMK, then GBH and ABH.

This construction is known as *applying* $FEGB$ to the given straight-line AB. The applied parallelogram has to fit angle $\angle BAL$.

Proposition I.45 summarizes a method we call the triangulation of polygons (see Fig. 1.69, left). Euclid's diagram depicts a quadrangle $ADCB$, nevertheless, the method applies to any polygon. The idea is this: divide the polygon $ADCB$ into adjacent triangles, say ADB, DCB;

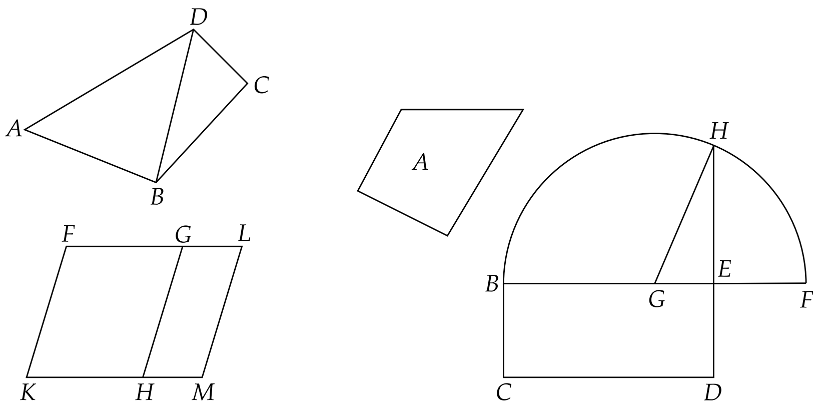


Figure 1.69: *Elements*, I.45 and II.14

by proposition I.41, transform each triangle into a parallelogram, say P_1, P_2; let us assume P_1 is simply $FGHK$; apply to the line GH a parallelogram equal to P_2. It easily follows that $FLMK = ADCB$. Then, the resulting parallelogram $FLMK$ is transformed into a rectangle. In this way, any polygon A can be transformed into an equal rectangle. The theory of equal figures culminates with a construction of squaring a rectangle introduced by proposition II.14 (see Fig. 1.69, right and an exposition of the construction as presented in (Błaszczyk, Mrówka, Petiurenko 2020).[15]

1.2.15 From equal figures to parallel lines. I 39–40

The theory of equal figures rests on the concept of a parallel line, meaning the unique line drawn according to I.31: all throughout propositions I.35–38, Euclid studies figures which are "between the same parallels". In I.39–40, he takes the other way around: starting from equal figures he seeks to reach parallel lines.

I.39 *Equal triangles which are on the same base, and on the same side, are also between the same parallels.*

Let $\triangle ABC = \triangle DBC$ (see Fig. 1.70, left). In I.8, Euclid shows that two equal, i.e., congruent, triangles on the same side of a line are not possible. With a new sense of equality,

[15]Euclid applies equal figures in the theory of proportion (Book V) and similar figures (Book VI). In modern foundations of geometry, there are many tries to recover Euclid's propositions I.35–45 based on non-Euclidean proportions, to mention (Hilbert 1899, ch. 4), (Hartshorne 2000, ch. 5), and (Beeson 2022).

they are possible.

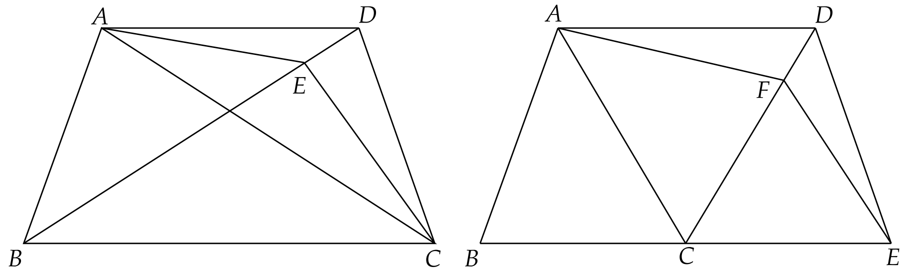


Figure 1.70: *Elements*, I.39, 40

For the proof, suppose $\triangle ABC = \triangle DBC$ and AD is not parallel to BC. Then, the construction part follows: the parallel to BC through A meets BD at E.

$$\frac{A \parallel BC, BD^{\rightarrow}}{E}$$

On the one hand, by I.37, $\triangle BEC = \triangle DBC$, on the other, $\triangle BEC < \triangle DBC$. *The very thing is impossible.*

$\square$

The proof of proposition I.40 reiterates the same *reductio ad absurdum* argument.

1.2.16 Pythagorean theorem. I 46–48

I.46 *To describe a square on a given straight-line.*

Proposition I.40 reiterates the same result in regard to triangles on equal bases.

	$A \perp AB$	$AC, (A, a)$	$D \parallel AB$	$B \parallel AD$	
A, B	C	D			E

The above table mirrors Euclid's construction. The last column highlights the fact that Euclid does not show that parallels to AB through the point D and to CD through B meet at all. Given that they meet, $ADEB$ is a parallelogram. Hence, by I.34, all of its sides are equal. Furthermore, since AD falls upon parallel lines AB, DE, and angle at A is right, then, by I.29, angle at D is also right.

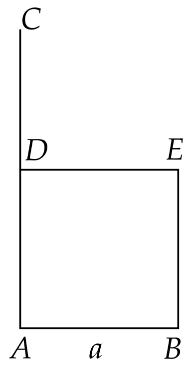


Figure 1.71: *Elements*, I.46

□

I.47 *In a right-angled triangle, the square on the side subtending the right-angle is equal to the squares on the sides surrounding the right-angle.*

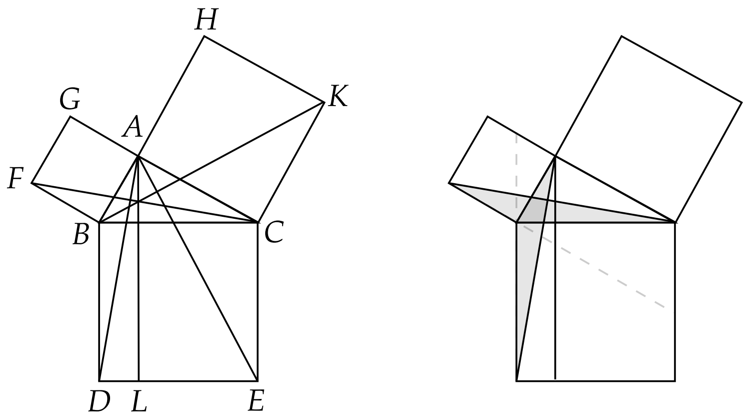


Figure 1.72: *Elements*, I.47 (left) and the crux step of its proof (right)

	sq on BC	*sq on* AB	*sq on* AC	$A \parallel BD,\, DE$
$A,\, B,\, C$	$D,\, E$	$F,\, G$	$H,\, K$	L

Euclid's construction includes arguments to the effect CA *is straight-on to* AG that may surprise a modern reader. Indeed, since AG is a leg of a right-angle triangle and CA is a side of a square described on another leg, he has to demonstrate these two line segments make a finite straight-line parallel to FB, another side of the square (see Fig. 1.72). Actually, it follows from I.15.

There are two proofs of the Pythagorean theorem in the *Elements*: I.47 and VI.31. The first relies on the theory of equal figures, the second – on similar figures. The former proceeds as explained below.

On the one hand, by SAS, $\triangle FBC = \triangle ABD$. By I.41

$$sq\, FGAB = 2\triangle FBC = 2\triangle ABD = rec\, DBL.$$

Similarly,

$$sq\, AHKC = rec\, CEL.$$

Hence

$$sq\, FGAB = sq\, AHKC = rec\, DBL + rec\, CEL = sq\, BCDE.$$

Wherein, the equality such as $rec\, DBL + rec\, CEL = sq\, BCDE$, is proved already in II.1 (Błaszczyk, Mrówka, Petiurenko 2020).

□

I.48 *If the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining sides of the triangle then the angle contained by the remaining sides of the triangle is a right-angle.*

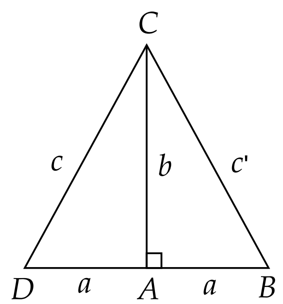


Figure 1.73: *Elements*, I.48; letters a, b, c, and c' added

	$A \perp AC$	$AD^{\rightarrow}$, Aa
A, C	D	B

In triangle $\triangle CAD$, the equality holds $c^2 = a^2 + b^2$, where $CD = c, AD = a, AC = b$. In the right triangle $\triangle CAB$, by I.47, the following equality holds $(c')^2 = a^2 + b^2$. And Euclid continues, "the square on DC is equal to the square on BC. So DC is also equal to BC". Hence, by SSS, $\triangle CAD = \triangle CAB$, and $\angle CAB = \pi/2 = \angle CAD$.

□

1.3 *Non-use of the Postulate* 5

Squarely in contrast to Euclidean practice, an opinion prevails that the Parallel Postulate is problematic, controversial, or unobvious. In proposition I.30 Euclid does not demonstrate the existence of the straight-line GK falling across three lines AB, EH, and CD (see Fig. 1.62). In I.39, he takes for granted that lines BC and the parallel to BC drawn through A meet in E (see Fig. 1.70).[16] In I.46, he does not show that the parallel to AB drawn through D and the parallel to AD drawn through B meet in E (see Fig. 1.71). In IV.5, when showing that the center of the circumcircle of a triangle is the intersection of perpendicular bisectors of its sides, he considers cases depending on whether they meet inside the triangle, or on its side, or outside it; he does not show that they meet at all. Indeed, in all these cases, the existence of intersection points follows from the Postulate 5. Interestingly enough, Euclid does not even mention there is a need to prove it. It is in contrast with the meticulousness of his other proofs.

Some of these propositions, e.g., IV.5, are equivalent forms of the Postulate 5; others, e.g., the relationship between I.30 and I.46, led to intriguing debate within the modern foundations of geometry (Pambuccian, Schacht 2021). In the subsequent section, we present a model of non-Euclidean plane and will discuss the assumptions underlying these propositions.

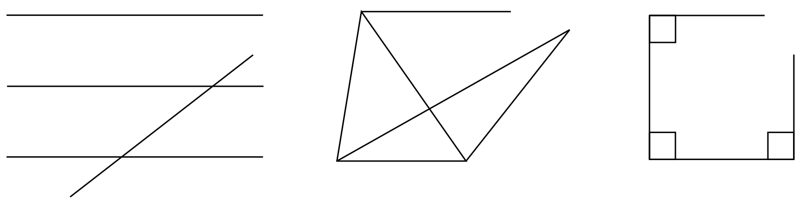


Figure 1.74: Seeking for intersection points in propositions I.30, 39, 46

[16]Similarly in constructions such as I.42, 44, 45.

1.4 Recent systems of Euclidean geometry

In this section, we present the Euclidean geometry systems developed in the 1950s and 1960s, which refer to Hilbert's axioms, namely (Borsuk, Szmielew 1960, Borsuk, Szmielew 1972, Tarski 1959, Tarski, Givant 1999).

1.4.1 Borsuk, Szmielew's axioms

K. Borsuk and W. Szmielew in (Borsuk, Szmielew 1960) as primitive notions take:

a) points; a, b, c, p, q

b) straight lines; K, L, M

d) ternary relation of *betweenness* $\mathbf{B}$. The formula $\mathbf{B}(a, b, c)$ stands for: *point b lies between points a and c*;

e) quaternary relation of *equidistance.* The formula $\mathbf{E}(a, b, c, d)$ stands for: *point a is as far from point b as point c is from point d.*

Sets of points are defined as a figures.

Axioms for plane geometry

I. Axioms of Incidence.

(I1.) $\forall L \exists a, b[a \neq b \wedge a, b \in L]$.

(I2.) $\forall a, b \exists L[a, b \in L]$.

(I3.) $\forall a, b \forall K, L[(a, b \in K \wedge a, b \in L) \rightarrow K = L]$.

(I4.) $\exists a, b, c[a \neq b \neq c \wedge \neg \exists L[a, b, c \in L]\,]$.

II. Linear Axioms of Order.

(O1.) $\mathbf{B}(abc) \rightarrow (a \neq b \neq c \wedge \exists L[a, b, c \in L]\,)$.

(O2.) $\mathbf{B}(abc) \rightarrow \mathbf{B}(cba)$.

(O3.) $\mathbf{B}(abc) \rightarrow \neg\mathbf{B}(bac)$.

(O4.) $\forall a, b, c \exists L[(a \neq b \neq c \wedge a, b, c \in L) \rightarrow (\mathbf{B}(abc) \vee \mathbf{B}(bca) \vee \mathbf{B}(cab))]$.

(O5.) $\forall a, b \exists c[a \neq b \rightarrow \mathbf{B}(abc)]$.

(O6.) $\forall a, b \exists c[a \neq b \rightarrow \mathbf{B}(acb)]$.

(O7.) $(\mathbf{B}(abc) \wedge \mathbf{B}(bcd)) \rightarrow \mathbf{B}(abd)$.

(O8.) $(\mathbf{B}(abd) \wedge \mathbf{B}(bcd)) \rightarrow \mathbf{B}(abc)$.

III. Pasch's axiom

(O9.) Given on a plane three non-collinear points a, b, c and a line K, if $\mathbf{B}(aKb)$ and $c \sim \in K$, to $\mathbf{B}(bKc)$ or $\mathbf{B}(aKc)$.

Where $c \sim \in K$ abbreviates $c \notin K$.

IV. Axioms of Congruence.

(C1.) $\mathbf{E}(aapq) \rightarrow p = q$.

(C2.) $\mathbf{E}(abba)$.

(C3.) $(\mathbf{E}(abpq) \wedge \mathbf{E}(abrs)) \rightarrow \mathbf{E}(pqrs)$.

(C4.) $(\mathbf{B}(a_1b_1c_1) \wedge \mathbf{B}(a_2b_2c_2)) \wedge a_1b_1 \equiv a_2b_2 \wedge b_1c_1 \equiv b_2c_2 \rightarrow a_1c_1 \equiv a_2c_2$.

(C5.) For every half-line A with origin a and for every segment pq there exists just one point $b \in A$ such that $ab \equiv pq$.

(C6.) Given lines L_1 and L_2 and points $a_1, b_1, c_1 \in L_1, d_1 \sim\in L_1, a_2, b_2, c_2 \in L_2, d_2 \sim\in L_2$, if $\mathbf{B}(a_1, b_1, c_1), \mathbf{B}(a_2, b_2, c_2), a_1b_1 \equiv a_2b_2, b_1c_1 \equiv b_2c_2, d_1a_1 \equiv d_2a_2$ and $d_1b_1 \equiv d_2b_2$, then $d_1c_1 \equiv d_2c_2$.

(C7.) Given a half-plane W with boundary K, a segment $ab \subset K$, and a triangle pqr, if $ab \equiv pq$, then there exists just one point $c \in W$ such that $ac \equiv pr$ and $bc \equiv qr$.

V. Axiom of Continuity

Given two arbitrary non-empty points sets X and Y, if there exists a point a such that

$p \in X$ and $q \in Y$ implies $\mathbf{B}(a, p, q)$,

then there exists a point b such that

$p \in X - b$ and $q \in Y - b$ implies $\mathbf{B}(p, b, q)$.

Where $X - b$ abbreviates $X \setminus \{b\}$.

1.4.2 Tarski's axioms

Tarski's Axioms aim to provide a rigorous basis for Euclidean geometry within the framework of first-order logic (Tarski 1959).

In the following:

$\equiv$ denotes the relation of equidistance. B denotes the relation of betweenness. $=$ denotes the relation of equality. The axioms are as follows:

Axiom 1: (Identity Axiom for Betweenness). $\forall x, y(\mathsf{B}xyx \implies x = y)$.

Axiom 2: (Transitivity Axiom for Betweenness). $\forall x, y, z, u[(\mathsf{B}xyu \wedge \mathsf{B}yzu) \implies \mathsf{B}xyz]$.

Axiom 3: (Connectivity Axiom for Betweenness). $\forall x, y, z, u[(\mathsf{B}xyz \wedge \mathsf{B}xyu \wedge x \neq y) \implies (\mathsf{B}xzu \vee \mathsf{B}xuz)]$.

Axiom 4: (Reflexivity Axiom for Equidistance). $\forall x, y(xy \equiv yx)$.

Axiom 5: (Identity Axiom for Equidistance). $\forall x, y, z(xy \equiv zz \implies x = z)$.

Axiom 6: (Transitivity Axiom for Equidistance). $\forall x, y, z, u, v, w[(xy \equiv zu \wedge xy \equiv vw) \implies zu \equiv vw]$.

Axiom 7: (Pasch Axiom) $\forall t, x, y, z, u[\exists v[\mathsf{B}xtu \wedge \mathsf{B}yuz \implies \mathsf{B}xvy \wedge \mathsf{B}ztv)]$.

Axiom 8: (Euclid's Axiom) $\forall t, x, y, z, u[\exists v, w((\mathsf{B}xut \wedge \mathsf{B}yuz \wedge (x \neq u)) \implies$
$(\mathsf{B}xzv \wedge \mathsf{B}xyw \wedge \mathsf{B}vtw)]]$.

Axiom 9: (Five-Segment Axiom). $\forall x, x', y, y', z, z', u, u'[((x \neq y) \wedge (\mathsf{B}xyz \wedge \mathsf{B}x'y'z')\wedge$
$(xy \equiv x'y' \wedge yz \equiv y'z' \wedge xu \equiv x'u' \wedge yu \equiv y'u'))] \implies zu \equiv z'u'$.

Axiom 10: (Axiom of Segment Construction). $\forall x, y, u, v(\exists z : \mathsf{B}xyz \wedge yz \equiv uv)$.

Axiom 11: (Lower dimension axiom). $\exists x, y, z(\neg\mathsf{B}xyz \wedge \neg\mathsf{B}yzx \wedge \neg\mathsf{B}zxy)$.

Axiom 12: (Upper dimension axiom). $\forall x, y, z, u, v[(xu \equiv xv \wedge yu \equiv yv \wedge zu \equiv zv \wedge u \neq v) \implies \mathsf{B}xyz \vee \mathsf{B}yzx \vee \mathsf{B}zxy]$.

Axiom 13: (Elementary continuity axioms). All sentences of the form $\forall vw : ...\exists z\forall xy[(\varphi \wedge \psi \implies \mathsf{B}zxy) \implies (\exists u[\forall xy[(\varphi \wedge \psi \implies \mathsf{B}xuy))]]]$ where φ stands for any formula in which the variables $x, v, w, ...$, but neither y nor z nor u, occur free, and similarly for ψ, with x and y interchanged.

In Table 1.1 we compare some solutions adopted by discussed systems. We will refer back to that table in Chapter 5 while reviewing secondary school textbooks.

Author	**Primitive concepts**	**Primitive relations**	**Algebra of line segments and angles**	**the Pasch axiom**	**SAS axiom**
Hilbert	point, line	betweenness, congruence of segments and angles	B1-B3, C2,C3,C5	line crossing sides of triangle.	C6+theorem 10
Euclid		greater-than addition	Common Notions		Proposition I.4
Hartshorne	point, line	betweenness, congruence of segments and angles	B1-B3, C2,C3, C5	line crossing sides of triangle, the cross-bar theorem, the plane separation theorem	C6
Borsuk, Szmielew	point, line	betweenness, equidistance	O.1-O.8, C1-C4, length of a line segment, measure of an angle	plane separation axiom	Five-Segment Axiom
Tarski	point	betweenness, equidistance	A1-A6	in terms of points and betweenness	Five-Segment Axiom

Table 1.1: Basic concepts of Euclidean geometry

1.5 Semi-Euclidean plane

In this section, we present a model of semi-Euclidean plane, i.e., a plane in which angles in a triangle sum up to π yet the parallel postulate fails. (Hartshorne 2000, 311), introduces that term, but the very idea originates in Max Dehn's 1900 (Dehn 1900, § 9), who built such a model owing to a non-Archimedean Pythagorean field. Dehn explored a non-Euclidean field introduced already in Hilbert (Hilbert 1899, § 12).[17] We employ the Euclidean field of hyperreal numbers. On the Cartesian plane over hyperreals, the circle-circle and circle-line intersection axioms are satisfied, meaning one can mirror Euclid's straightedge and compass constructions. To elaborate, let us start with the introduction of the hyperreal numbers.

[17] See also (Hartshorne 2000, § 18). Example 18.4.3 expounds Dehn's model.

1.5.1 Cartesian Plane over an Ordered Field

A commutative field $(\mathbb{F}, +, \cdot, 0, 1)$ together with a total order $<$ on $\mathbb{F}$ forms an ordered field when field operations are compatible with the order, that is

$$x < y \Rightarrow x + z < y + z, \quad x < y,\, 0 < z \Rightarrow xz < yz.$$

In every ordered field, one can define the absolute value

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0, \end{cases}$$

and the following subsets of $\mathbb{F}$:

$$\begin{aligned} \mathbb{L} &= \{x : (\exists n \in \mathbb{N})(|x| < n)\}, \\ \Psi &= \{x : (\forall n \in \mathbb{N})(|x| > n)\}, \\ \Omega &= \{x : (\forall n \in \mathbb{N})(|x| < \tfrac{1}{n})\}. \end{aligned}$$

The elements of these sets we call finite, infinitely large, and infinitely small (infinitesimals) numbers, respectively.

Here are some obvious relationships between these kinds of elements,

$$\begin{aligned} &(\forall x, y \in \Omega)(x + y \in \Omega, xy \in \Omega), \\ &(\forall x \neq 0)(x \in \Omega \Leftrightarrow\ x^{-1} \in \Psi), \\ &(\forall x \in \Omega)(\forall y \in \mathbb{L})(xy \in \Omega). \end{aligned} \tag{1.1}$$

An ordered field is a Pythagorean field iff every sum of two squares is a square: that is

$$(\forall a, b \in \mathbb{F})(\exists c \in \mathbb{F})(a^2 + b^2 = c^2).$$

An ordered field is a Euclidean field iff every non-negative element is a square: that is

$$(\forall a \in \mathbb{F}_+)(\exists b \in \mathbb{F})(a = b^2), \quad \text{where} \quad \mathbb{F}_+ = \{x \in \mathbb{F} : x \geq 0\}.$$

An ordered field is a Archimedean field iff one of the following statements holds:

(A1) $(\forall x, y \in \mathbb{F})(\exists n \in \mathbb{N})(0 < x < y \Rightarrow nx > y)$,

(A2) $(\forall x \in \mathbb{F})(\exists n \in \mathbb{N})(n > x)$, (A3) $\Omega = \{0\}$.

Primitive concepts of the Hilbert system – point, straight line, plane, relation of betweenness, congruence of line segments and congruence of angles – we interpret on the Cartesian plane $\mathbb{F} \times \mathbb{F}$ as follows:

(0) A plane is the set $\mathbb{F} \times \mathbb{F}$.

(1) A point is an ordered pair (a, b), where $a, b \in \mathbb{F}$.

(2) A straight line l line is the set of point defined by a linear equation $ax + by + c = 0$, with $a^2 + b^2 \neq 0$ and $a, b \in \mathbb{F}$; $l = \{(x, y) \in \mathbb{F} \times \mathbb{F} : ax + by + c = 0\}$.

(3) A circle is an object defined by quadratic equation $(x - a)^2 + (y - b)^2 = r^2$, where $a, b, r \in \mathbb{F}$; $\{(x, y) \in \mathbb{F} \times \mathbb{F} : (x - a)^2 + (y - b)^2 = r^2\}$.

We introduce the concept of circle due to the Euclid system.

(4) An angle is the union of two rays emanating from a point and not lying on the same line.

Let us emphasize that all the coefficients in the above formulas are elements in $\mathbb{F}$.

In the plane $\mathbb{F} \times \mathbb{F}$, a distance between two points $A = (a_1, a_2)$, $B = (b_1, b_2)$ is defined by:

$$|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \tag{1.2}$$

When the field $\mathbb{F}$ is Euclidean or Pythagorean, $|AB|$ belongs to $\mathbb{F}$; we call it the length of the segment AB. *Segments are congruent* if their lengths are equal.

We will use the definition of a tangent of angle to compare triangles:

If α is an angle formed by two rays r, r' lying on lines with slopes m, m', then the tangent of α is defined by

$$\tan \alpha = \pm \left| \frac{m' - m}{-1 + mm'} \right| . \tag{1.3}$$

where we take + if the angle is acute and – if the angle is obtuse. We also adopt the following convention $\tan \pi/2 = \infty$ (see (Hartshorne 2000), 142).

We define *the congruence of angles* as the equality of the respective tangents. Note that $\tan \alpha \in \mathbb{F} \cup \{\infty\}$.

We say that two *triangles are congruent* on the Cartesian plane $\mathbb{F} \times \mathbb{F}$ if there exists a rigid motion that transforms one triangle onto another.

For the interpretation of the betweenness on the Cartesian $\mathbb{F} \times \mathbb{F}$ plane, we use the fact that $\mathbb{F}$ is an ordered field. For example (see Fig. 1.75): let points $A = (a_1, a_2)$, $B = (b_1, b_2$, $C = (c_1, c_2)$ lie on the same line. We project these points on the Ox axis. Then $A * B * C$ holds when $a_1 < b_1 < c_1$.

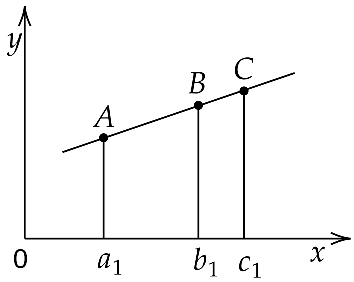


Figure 1.75: Betweenness in the Cartesian plane $\mathbb{F} \times \mathbb{F}$

Hartshorne (Hartshorne 2000, ch. 3), shows that with such an interpretation of primitive concepts, Hilbert's axioms from the first four groups are satisfied.

Definition 1.5.1. The Hilbert plane is the plane in which the axioms of incidence, beetweeness, and congruence are satisfied.

The following theorem summarizes the above considerations:

Theorem 1.5.2. *The Cartesian plane* $\mathbb{F} \times \mathbb{F}$ *is the Hilbert plane if and only if the field* $\mathbb{F}$ *is a Pythagorean.*

There is no circle among primitive concepts in the Hilbert system, nevertheless, it can be defined. Yet, generally, in the Hilbert plane, circles do not intersect. In the *Elements*, points are often introduced by an intersection of circles, that is why we introduce the following axiom:

Circle-circle intersection axiom

Given two circles O_1 and O_2, if O_2 contains at least one point inside O_1, and O_2 contains at least one point outside O_1, then O_1 and O_2 will meet. (Hartshorne 2000, 108)

Definition 1.5.3. The Euclidean plane is the Hilbert plane with additional circle-circle intersection axiom.

Now we can formulate an appropriate theorem for the Euclidean plane:

Theorem 1.5.4. *The Cartesian plane $\mathbb{F} \times \mathbb{F}$ is the Euclidean plane if and only if the field F is Euclidean.(Hartshorne 2000, 142)*

In section §1.5.3, we will introduce a Cartesian plane over a Euclidean (non-Archimedean) field of hyperreal numbers.

1.5.2 Hyperreal numbers

We define the set of hypperreals as the quotient set on the set of all sequences of real numbers, $\mathbb{R}^{\mathbb{N}}$, with respect to a specific relation defined on the set of indexes $\mathbb{N}$. To this end, we need a notion of ultrafilter on $\mathbb{N}$.[18]

A family of sets $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ is an ultrafilter on $\mathbb{N}$ iff (1) $\emptyset \notin \mathcal{U}$, (2) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, (3) if $A \in \mathcal{U}$ and $A \subset B$, then $B \in \mathcal{U}$, (4) for each $A \subset \mathbb{N}$, either A or its complement $\mathbb{N} \setminus A$ belongs to $\mathcal{U}$.

The family of sets with finite complements satisfies conditions (1)–(3) listed in the definition of an ultrafilter. By Zorn's lemma, this family can be extended to an ultrafilter. Let $\mathcal{U}$ denote a fixed ultrafilter on $\mathbb{N}$ containing every subset with a finite complement.[19]

Since $\mathbb{N}$ and sets of the form $\{k, k+1, k+2, ...\}$, in short $\{n \in \mathbb{N} : n \geq k\}$, for every $k \in \mathbb{N}$, comply with the proviso "have finite complements", they all belong to the ultrafilter $\mathcal{U}$,

$$\mathbb{N} \in \mathcal{U}, \quad \{n \in \mathbb{N} : n \geq k\} \in \mathcal{U}, \quad \text{for every } k.$$

These sets will do to check examples concerning infinitesimals and infinite numbers, which we present below.

In the set $\mathbb{R}^{\mathbb{N}}$ we define an equivalence relation by

$$(r_n) \equiv (s_n) \Leftrightarrow \{n \in \mathbb{N} : \ r_n = s_n\} \in \mathcal{U}.$$

Let $\mathbb{R}^*$ denote the reduced product $\mathbb{R}^{\mathbb{N}}/_{\equiv}$. Clearly, the equality of hyperreals is defined by

$$[(r_n)] = [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : \ r_n = s_n\} \in \mathcal{U}. \tag{1.4}$$

[18] This presentation follows (Goldblatt, 1998), (Błaszczyk, Petiurenko 2022).

[19] In research papers, such an ultrafilter is called non-principal. A principal ultrafilter contains a singleton $\{k\}$, for some $k \in \mathbb{N}$. As a result the ultraproduct $\mathbb{R}^{\mathbb{N}}/_{\equiv}$ contains no *new* elements, i.e., is not an extension of $\mathbb{R}$.

Since $\mathfrak{U}$ is the ultrafilter, we also get the following condition

$$[(r_n)] \neq [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : \; r_n \neq s_n\} \in \mathfrak{U}.$$

To illustrate how it works, let us observe that an inequality

$$[(0, 0, 0, ...)] \neq [(r_1, r_2, r_3, ...)]$$

translates into condition $\{n \in \mathbb{N} : r_n \neq 0\} \in \mathfrak{U}$, rather than $\{n \in \mathbb{N} : r_n \neq 0\} = \mathbb{N}$. The above condition is used, i.a., to determine the multiplicative inverse of a hyperreal, namely

$$[(r_n)]^{-1} = [(r_1^{-1}, r_2^{-1}, r_3^{-1}, ...)], \quad \text{for} \quad [(r_n)] \neq [(0, 0, 0, ...)].$$

Although some elements of the sequence (r_j) can be zero, the requirement $[(r_n)] \neq [(0, 0, 0, ...)]$ translates into $\{j \in \mathbb{N} : r_j \neq 0\} \in \mathfrak{U}$, meaning $r_j = 0$, for $j \in \mathbb{N} \setminus \{j \in \mathbb{N} : r_j \neq 0\}$. Since $\{j \in \mathbb{N} : r_j \neq 0\} \in \mathfrak{U}$, it follows that $\mathbb{N} \setminus \{j \in \mathbb{N} : r_j \neq 0\} \notin \mathfrak{U}$, or equivalently, $\{j \in \mathbb{N} : r_j = 0\} \notin \mathfrak{U}$. From the perspective of the definition 1.4, elements of the sequence (r_j) such that $r_j = 0$ does not affect the hyperreal number $[(r_j)]$.

Now, we can define a new sequence (s_j) as follows

$$s_j = \begin{cases} r_j^{-1}, & \text{if} \qquad j \in \{j \in \mathbb{N} : r_j \neq 0\}, \\ 0, & \text{if} \;\; j \in \mathbb{N} \setminus \{j \in \mathbb{N} : r_j \neq 0\}, \end{cases}$$

Therefore, strictly speaking

$$[(r_n)]^{-1} =_{df} [(s_j)].$$

The above procedure illustrates the claim that to define a hyperreal number $[(r_j)]$, one has to determine r_j for $j \in A$, for some $A \in \mathfrak{U}$, rather than for all $j \in \mathbb{N}$.

New sums and products are defined pointwise, that is

$$[(r_n)] + [(s_n)] = [(r_n + s_n)], \quad [(r_n)] \cdot [(s_n)] = [(r_n \cdot s_n)].$$

A total order on $\mathbb{R}^*$ is defined by

$$[(r_n)] < [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : r_n < s_n\} \in \mathfrak{U}.^{20}$$

[20] In these definitions, we adopt a standard convention to use the same signs for the relations and operations on $\mathbb{R}$ and $\mathbb{R}^*$.

A standard real number $r \in \mathbb{R}$ is represented by the class $[(r, r, r, ...)]$. In what follows, we will use the letter r for hyperreal number $[(r, r, r, ...)]$.

By a straightforward checking, we find that $(\mathbb{R}^*, +, \cdot, 0, 1, <)$ is an ordered field.

The field of real numbers is the *biggest* Archimedean field, meaning every Archimedean field is isomorphic to some subfield of real numbers (see (Hartshorne 2000), 139). Since the equality $\Omega = \{0\}$ is equivalent to the Archimedean axiom, a single non-zero infinitesimal will do to prove that hyperreals extend the field of real numbers. To this end, we will show that the hyperreal number $\varepsilon = [(1, \frac{1}{2}, \frac{1}{3}, ...)]$ is positive and smaller than any positive real number, i.e.,

$$[(1, \tfrac{1}{2}, \tfrac{1}{3}, ...)] < [(r, r, r, ...)], \quad \text{for every} \quad r \in \mathbb{R}_+.$$

For a proof, take $r > 0$. In real analysis, $(1/n)$ is a null-sequence, i.e., $\lim\limits_{n\to\infty} 1/n = 0$.[21] By the definition of the limit of a sequence, it means there is an index k such that $1/n < r$, for all $n > k$. In terms of the elements of the ultrafilter, it means $\{n \in \mathbb{N} : n > k\} \in \mathfrak{U}$, or

$$\{n \in \mathbb{N} : \tfrac{1}{n} < r\} \in \mathfrak{U}.$$

Due to the definition of the total order in $\mathbb{R}^*$, it means

$$[(1, \tfrac{1}{2}, \tfrac{1}{3}, ...)] < [(r, r, r, ...)].$$

By the same argument, i.e., based on the fact $\lim\limits_{n\to\infty} 1/n = 0$, we can show that definitions of the sets Ω, Ψ, and $\mathbb{L}$, in the field of hyperreals translate into the following ones

$$\begin{aligned} \Omega &= \{x \in \mathbb{R}^* : (\forall r \in \mathbb{R}_+)(|x| < r)\}, \\ \Psi &= \{x \in \mathbb{R}^* : (\forall r \in \mathbb{R})(|x| > r)\}, \\ \mathbb{L} &= \{x \in \mathbb{R}^* : (\exists n \in \mathbb{R})(|x| < r)\}, \end{aligned}$$

given that r stands for the hyperreal number $[(r, r, r, ...)]$.

Thus $\varepsilon \in \Omega$. The hyperreal

$$\varepsilon^2 = [(1, \tfrac{1}{2^2}, \tfrac{1}{3^2}, ...)]$$

is another infinitesimal. Generally, if (r_n) is a null-sequence, then $[(r_n)]$ is infinitesimal.[22]

---

[21] Indeed, it is yet another form of the Archimedean axiom.

[22] See (Błaszczyk, Major, 2014).

Since $\varepsilon \in \Omega$, then $\varepsilon^{-1} = [(1, 2, 3, ...)]$ is an infinitely large number. Indeed, $N = [(1, 2, 3, ..)]$ exemplifies infinite number. Other infinite numbers are

$$N + 1 = [(2, 3, 4, ...)], \quad N^2 = [(1, 4, 9, ...)], \quad N! = [(1!, 2!, 3!, ..)].$$

Fig. 1.76 represents in a schematized way a relationship between $\mathbb{R}$ and $\mathbb{R}^*$, as well as between $\mathbb{L}$, Ψ, and Ω.

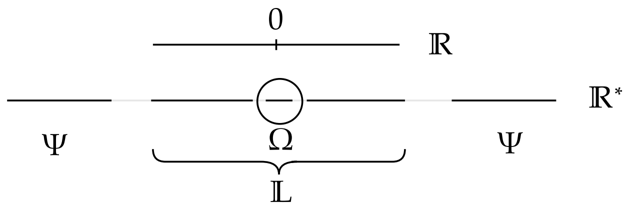


Figure 1.76: The line of real numbers and its extension to hyperreals

1.5.3 Cartesian Plane over the Hyperreals

Let f be a real map, i.e. $f : \mathbb{R} \mapsto \mathbb{R}$. By f^* we mean a map $f^* : \mathbb{R}^* \mapsto \mathbb{R}^*$ defined by

$$f^*([(r_n)]) = [(f(r_1), f(r_2), ...)]. \tag{1.5}$$

If $r \in \mathbb{R}$, then $f^*(r) = [(f(r), f(r), ...)]$. Since we identify real number r with hyperreal $[(r, r, ...)]$, the equality $f^*([(r, r, ...)]) = f(r)$ obtains, meaning $f^*_{|\mathbb{R}} = f$.

Putting $f = \sqrt{}$ in definition 1.5, we get

$$\sqrt{[(r_n)]}^* = [(\sqrt{r_n})] = [\sqrt{r_1}, \sqrt{r_2}, ...)], \quad \text{for} \quad [(r_j)] > 0.$$

It proves that the field of hyperreals is Euclidean, and the interpretation of the congruence of line segments based on (1.2) applies to the Cartesian plane $\mathbb{R}^* \times \mathbb{R}^*$.

Similarity, under the definition 1.5, we have

$$\sin^*[(r_n)] = [(\sin r_1, \sin r_2, ...)], \quad \cos^*[(r_n)] = [(\cos r_1, \cos r_2, ...)].$$

Since for every n the identity $\sin^2 r_n + \cos^2 r_n = 1$ holds, we have

$$(\sin^* x)^2 + (\cos^* x)^2 = 1.$$

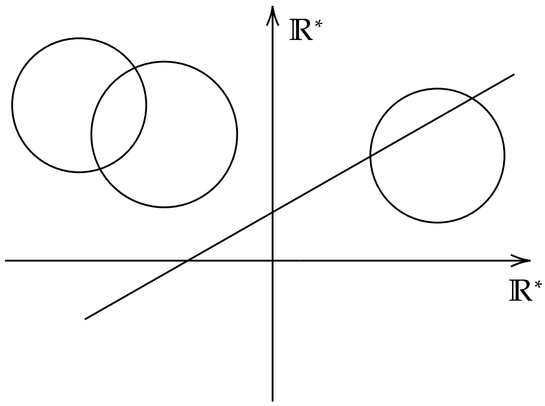


Figure 1.77: Cartesian plane over the field of hyperreals

Similarly, every trigonometric identity can be transferred into an identity involving the maps $\sin^*$ and $\cos^*$, and $\tan^*$. Consequently, the interpretation of the congruence of angles based on (1.3) applies to the Cartesian plane $\mathbb{R}^* \times \mathbb{R}^*$, although we could also simplify it and consider the map $\tan^*$.

Summing up, the Cartesian plane over the field of hyperreal numbers is a model of Euclidean plane, with straight lines and circles given by equations $ax+by+c=0$, $(x-a)^2+(y-b)^2=r^2$, where $a, b, c, r \in \mathbb{R}^*$. Specifically triangles in the plane $\mathbb{R}^* \times \mathbb{R}^*$ satisfy the law that angles sum up to π. Parallel lines are of the form $y = mx + b$ and $y = mx + c$, while a perpendicular to the line $y = mx + b$ is given by the formula $y = \frac{-1}{m}x + d$. It is a non-Archimedean plane, specifically it includes straight lines determined by an infinitesimal slope, $y = \varepsilon x$, $\varepsilon \in \Omega$, as well as infinitely large slope $y = Nx$, with $N \in \Psi$.

1.5.4 Semi-Euclidean plane $\mathbb{L} \times \mathbb{L}$

In this section, we consider a subspace $\mathbb{L} \times \mathbb{L}$ of the plane $\mathbb{R}^* \times \mathbb{R}^*$. On that plane, circles are defined by analogous formula, namely $(x-a)^2 + (y-b)^2 = r^2$, where $a, b, c, r \in \mathbb{L}$, while every line in $\mathbb{L} \times \mathbb{L}$ is of the form $l \cap \mathbb{L} \times \mathbb{L}$, where l is a line in $\mathbb{R}^* \times \mathbb{R}^*$. The reason for that motion is as follows: since we want plane $\mathbb{L} \times \mathbb{L}$ include lines such as $y_1 = \varepsilon x$, where $\varepsilon \in \Omega$, it has also to include the perpendiculars $y_2 = \frac{-1}{\varepsilon}x$, but $\frac{-1}{\varepsilon} \notin \mathbb{L}$. Formula $l \cap \mathbb{L} \times \mathbb{L}$, where $l = ax + by + c$ and $a, b, c \in \mathbb{R}^*$ guarantees the existence of the straight lines with infinitely large slopes, such as y_2 in $\mathbb{L} \times \mathbb{L}$. The congruence of angles gets the same interpretation as in the model $\mathbb{R}^* \times \mathbb{R}^*$.

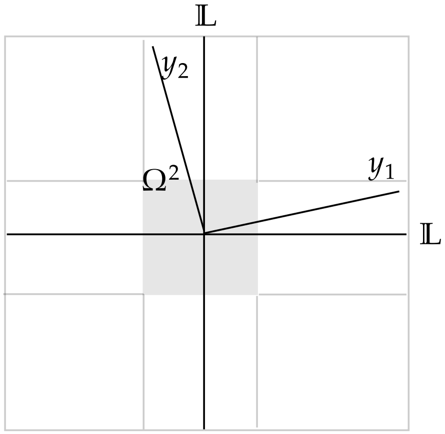


Figure 1.78: Perpendicular lines with infinitesimal and infinitely large slopes

Explicit checking shows that the model characterized above satisfies all Hilbert axioms of plane geometry plus the circle-circle and line-circle axioms, except parallel axiom.

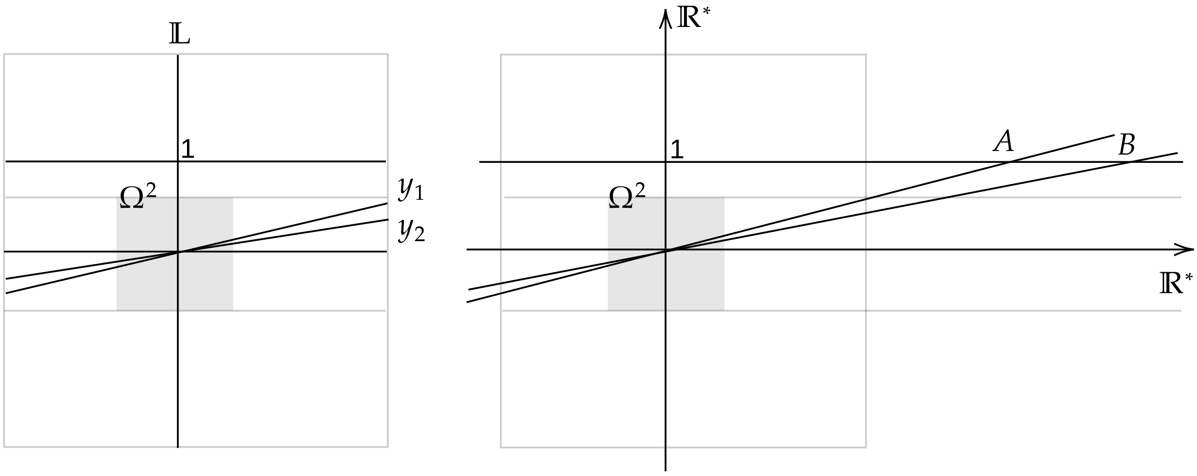


Figure 1.79: Non-Euclidean plane $\mathbb{L} \times \mathbb{L}$ (left) vs. Euclidean plane $\mathbb{R}^* \times \mathbb{R}^*$ (right)

Turning to the parallel postulate, we will show that infinitely many straight lines go through point $(0, 0)$ and do not intersect the line $y = 1$ (see Fig. 3.3). Let start with specific two straight lines: $y_1 = \varepsilon x, y_2 = \delta x$, where $\varepsilon, \delta \in \Omega$. Since $\Omega\mathbb{L} \subset \Omega$, the following inclusions hold $y_1, y_2 \subset \mathbb{L} \times \Omega$. In other words, values of maps $y_1(x), y_2(x)$ are infinitesimals, given that $x \in \mathbb{L}$. The same obtains for any line of the form $y = \mu x$, with $\mu \in \Omega$. Since there are infinitely many infinitesimals, there are infinitely many lines through $(0,0)$ not intersecting the horizontal line $y=1$.

To put it in Euclidean context: line $x=0$ falls on y_1 and $y=1$ making internal angles less that π, yet y_1 and $y=1$ do not intersect being produced to infinity in $\mathbb{L}\times\mathbb{L}$. Nevertheless, considering the plane $\mathbb{R}^*\times\mathbb{R}^*$, line $y=1$ meets y_1 at point $A=(\frac{1}{\varepsilon},1)$ and y_2 at $B=(\frac{1}{\delta},1)$.

Finally, the claim that angles in a triangle sum up to π follows from the theorem that $\mathbb{R}^*\times\mathbb{R}^*$ meets that law and observation that every triangle in $\mathbb{L}\times\mathbb{L}$ is a triangle in $\mathbb{R}^*\times\mathbb{R}^*$.

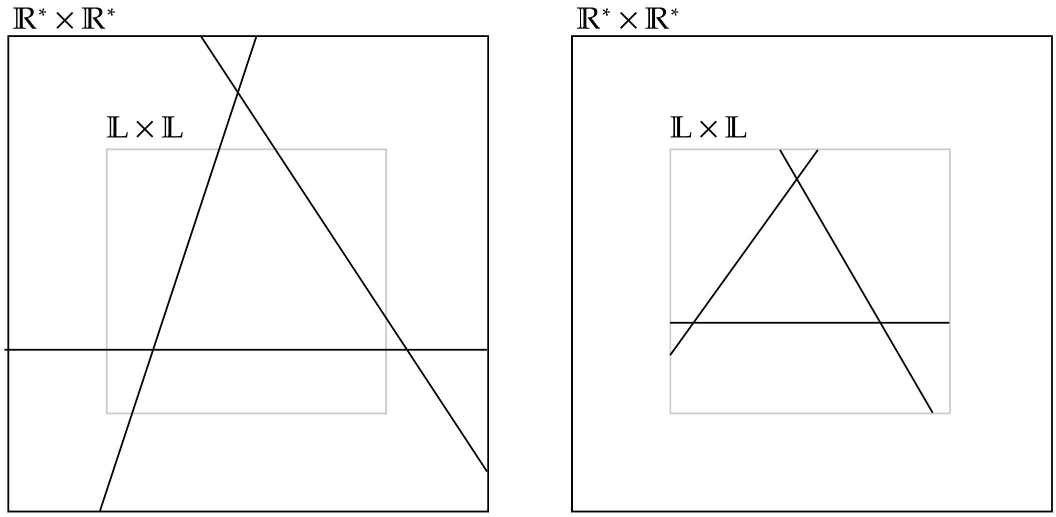


Figure 1.80: A triangle in Euclidean plane $\mathbb{R}^*\times\mathbb{R}^*$ (left). A triangle in $\mathbb{L}\times\mathbb{L}$ (right)

1.5.5 Euclid's propositions which do not hold in the plane $\mathbb{L}\times\mathbb{L}$

In proposition I.31, Euclid assumes the existence of a line falling on three parallel lines. In the plane $\mathbb{L}\times\mathbb{L}$ that assumption fails. Let us take three horizontal lines $y_1=0$, $y_2=\varepsilon$, $y_3=1$. Line $y_4=\varepsilon x$ meets y_1 at $(0,0)$ and y_2 at $(1,\varepsilon)$. Yet the intersection of y_4 and y_1 is $(\frac{1}{\varepsilon},1)$ which does not belong to $\mathbb{L}\times\mathbb{L}$ (see Fig 1.81).

In I.46, describing a square on a given straight line, Euclid draws parallels to sides of a right angle. As the construction applies I.31, they are perpendicular to the sides of the right angle. In the below examples, we take them to be literally parallels. In the plane $\mathbb{L}\times\mathbb{L}$, they have to

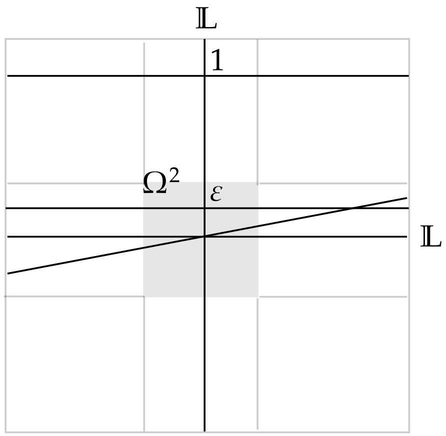


Figure 1.81: Transverse line crossing two of the three parallel lines.

meet, but the resulting figure is not a square (see Fig 1.82).

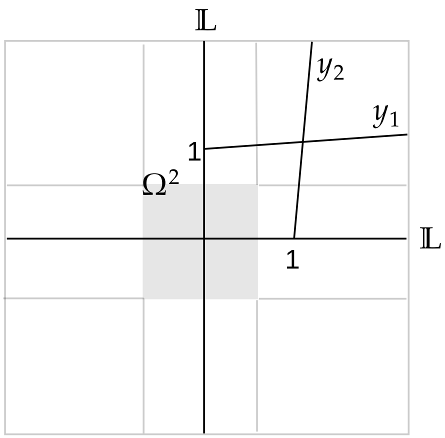


Figure 1.82: Parallels to sides of the right angle.

Let $y_1 = \varepsilon x + 1$ and $y_2 = \frac{x}{\varepsilon} - \frac{1}{\varepsilon}$, where $\varepsilon \in \Omega$. They do not intersect $y = 0$ and $x = 0$, respectively. Yet, they meet at the point $(\frac{1+\varepsilon}{1+\varepsilon^2}, 1+\frac{\varepsilon^2+\varepsilon}{1+\varepsilon^2})$, and the resulting figure is not a square.

In VI.5, Euclid assumes that perpendicular bisectors of two sides of a triangle meet. Below we show it does not hold in the plane $\mathbb{L} \times \mathbb{L}$.

Let us take the line $y = -\varepsilon$ and points $A = (-1, -\varepsilon)$, $B = (1, -\varepsilon)$ on it. A line through the point $C = (0, 0)$ and A has the equation $y = \varepsilon x$. Perpendicular bisectors of the sides AB and AC have the equations $x = 0$ and $2x + 2\varepsilon y + \varepsilon^2 + 1 = 0$, respectively. They meet at the point $\big(0, -\varepsilon - \frac{1}{\varepsilon}\big)$. However, $\big(0, -\varepsilon - \frac{1}{\varepsilon}\big) \notin \mathbb{L} \times \mathbb{L}$. Fig. 1.83 depicts three perpendiculars to the sides of the triangle ABC.

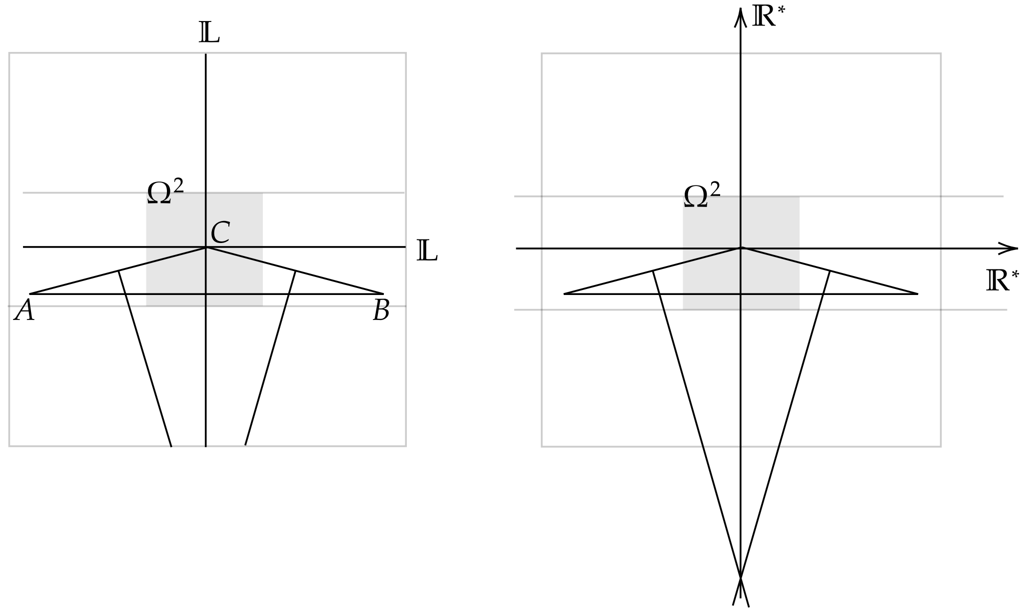


Figure 1.83: Triangle in $\mathbb{L}\times\mathbb{L}$ with no circumcircle (left) and its counterpart in $\mathbb{R}^*\times\mathbb{R}^*$ (right).

1.5.6 Klein and Poincaré disks

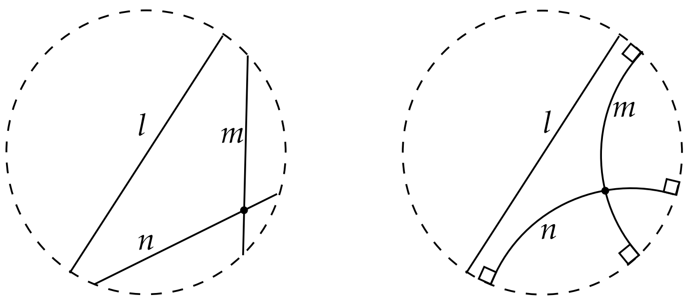


Figure 1.84: Straight line in Klein (left) and Poincaré (right) disk

Klein and Poincaré disks are classical models of non-Euclidean geometry (see Fig 1.84). Both consist of a fixed circle in the Euclidean plane, say Γ, representing the plane. In the Klein disk, chords of Γ are straight lines; in the Poincaré disk, straight lines are diameters of Γ or arcs of circles orthogonal to Γ (see Fig. 1.84).

In the Poincaré model, an angle between intersecting circles is the Euclidean angle between tangents to these lines drawn at their intersection point (see Fig 1.85).

In the Klein disc, an angle between intersecting straight lines is retrieved from the Poincaré model as presented in Fig. 1.86: for lines n, m, we draw circles orthogonal to Γ and determine

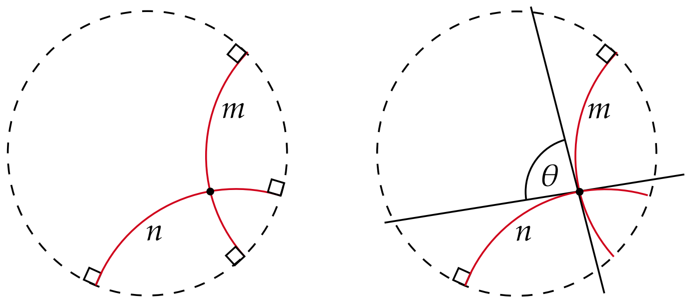


Figure 1.85: Angles in the Poincaré disc

the angle between them.

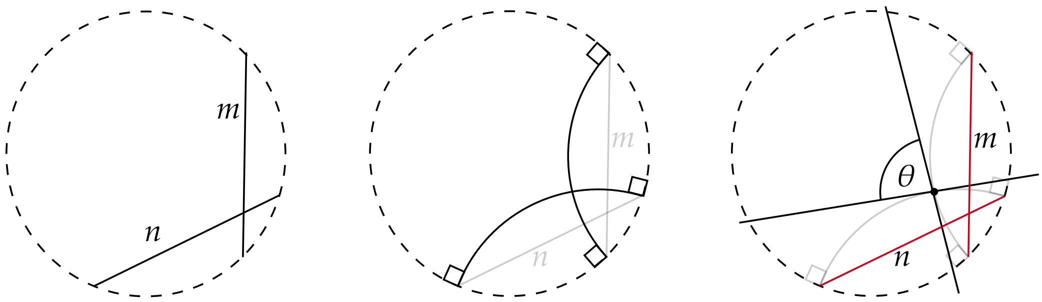


Figure 1.86: Angles in the Klein disc

Standard models of the non-Euclidean plane change Euclid's concept of a straight line or angle. In the plane $\mathbb{L} \times \mathbb{L}$, they are both Euclidean. Moreover, we can also develop Euclidean trigonometry in that plane. From a *local* perspective, thus, the plane $\mathbb{L} \times \mathbb{L}$ is *similar* to a Euclidean plane, *globally* – to put it metaphorically – its straight lines are *too short* to meet the Parallel Postulate. Last but not least, that plane has a unique educational advantage: expounding crucial ideas of that model requires only the basics of Cartesian geometry and non-Archimedean fields.

Chapter 2

Thales' Theorem through the 20th-century Foundations of Geometry

In this chapter, we present the 20th-century proofs of Thales' theorem and show that for this purpose, modern systems develop techniques beyond those described in Chapter 1. These are the arithmetic of line segments – the idea originated in Descartes' *Geometry* (1637) – or the arithmetic of real numbers, amplified by a version of the continuity axiom.

2.1 Hilbert's and Harsthorne's systems

In (Hilbert 1899), Thales' theorem is the final one in the chapter entitled Theory of Proportion. Hilbert starts with the sentence: "At the beginning of this chapter, we shall present briefly certain preliminary ideas concerning complex number systems which will later be of service to us in our discussion." It then lists 17 properties of numbers relating to addition, multiplication, relations $<, >$.

In the following, Hilbert makes it clear that: "In the present chapter, we propose, by aid of these axioms, to establish Euclid's theory of proportion; that is, we shall establish it for the plane and that independently of the axiom of Archimedes".

We then follow the long and arduous proof of a special case of Pascal's theorem:

Theorem 2.1.1 (Pascal's theorem.)**.** *Given the two sets of points A, B, C and A', B', C' so situated respectively upon two intersecting straight lines that none of them fall at the intersection* *of these lines. If CB' is parallel to BC' and CA' is also parallel to AC', then BA' is parallel to AB'.*

The proof of Pascal's theorem is quite complicated and Hilbert devotes all subsection to it. But the title of the next section "Algebra of segments, Based on Pascal's Theorem" suggests that the above theorem and its proof are very important for segment arithmetic.

Definition 2.1.2. If A, B, C are three points on a straight line and if B lies between A and C, then we say that $c = AC$ is the sum of the two segments $a = AB$ and $b = BC$. We indicate this by writing $c = a + b$.

Properties of addition:

1. The segments a and b are said to be smaller than c, this fact we indicate by writing $a < c$, $b < c$. On the other hand, c is said to be larger than a and b, and we indicate this by writing $c > a$, $c > b$.

2. For the definition 2.1.2 of addition of segments, the associative law is valid: $(a+b)+c = a + (b + c)$.

3. For the definition 2.1.2 of addition of segments, the commutative law is valid: $a+b = b+a$.

The properties of addition of segments can be proved based on the Hilbert axioms system.

Before we define product of two segment, Hilbert had selected any convenient segment, which, having been selected, shall remain constant throughout the discussion, and denote the same by 1.

Hilbert defines multiplication of segments as follows:

Definition 2.1.3. Upon the one side of a right angle, lay off from the vertex O the segment 1 and also the segment b. Then, from O lay off upon the other side of the right angle the segment a. Join the extremities of the segments 1 and a by a straight line, and from the extremity of b draw a line parallel to this straight line. This parallel will cut off from the other side of the right angle a segment c. We call this segment c the product of the segments a and b, and indicate this relation by writing $c = ab$ (see Fig 2.1).

Hilbert demonstrates that, for his the product of segments, the commutative, associative and distributive laws are held. Hilbert uses Pascal's theorem to prove the mentioned properties of the multiplication of segments.

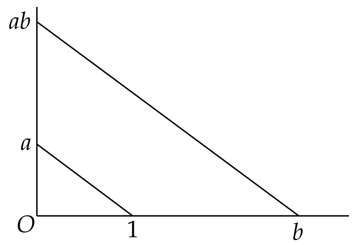


Figure 2.1: Definition of product of segments by Hilbert

Hilbert introduces the concept of division of segments, the definition of proportion and the similar triangles.

If b and c are any two arbitrary segments, there is always a segment a to be found such that $c = ab$. This segment a is denoted by $\frac{c}{b}$ and is called the quotient of c by b.

Definition 2.1.4. If a, b, a', b' are any four segments whatever, the proportion $a : b = a' : b'$ expresses nothing else than the validity of equation $ab' = ba'$.

Definition 2.1.5. Two triangles are called similar when the corresponding angles are congruent.

We already have a lot to formulate Thales' theorem, but not everything to prove it. Knowing this, Hilbert introduces an auxiliary theorem.

Theorem 2.1.6. *If a, b and a', b' are homologous sides of two similar triangles, we have the proportion $a : b = a' : b'$*

Proof. We shall first consider the special case where the angle included between a and b and the one included between a' and b' are right angles. Moreover, we shall assume that the two triangles are laid off in one and the same right angle. Upon one of the sides of this right angle, we lay off from the vertex 0 the segment 1, and through the extremity of this segment, we draw a straight line parallel to the hypotenuses of the two triangles (see Fig. 2.2).

This parallel determines upon the other side of the right angle a segment e. Then, according to our definition of the product of two segments, we have $b = ea$, $b' = ea'$, from which we obtain $ab' = ba'$, that is to say, $a : b = a' : b'$.

Let us now return to the general case. In each of the two similar triangles, find the point of intersection of the bisectors of the three angles. Denote these points by S and S'. From thesepoints let the perpendiculars r and r' fall upon the sides of the triangles, respectively.

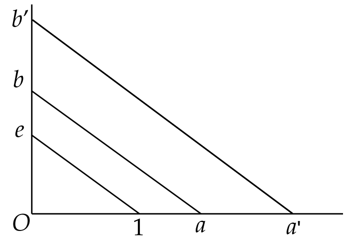


Figure 2.2: Proposition 2.1.6

Denote the segments thus determined upon the sides of the triangles by a_b, a_c, b_c, b_a, c_a, c_b and a'_b, a'_c, b'_c, b'_a, c'_a, c'_b respectively.

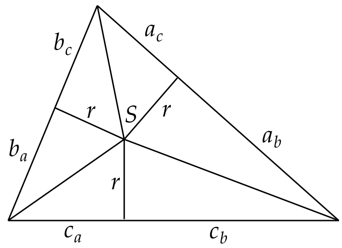


Figure 2.3: Proposition 2.1.6

The special case of our proposition, demonstrated above, gives us the following proportions:

$$a_b : r = a'_b : r', \;\; b_c : r = b'_c : r',$$

$$a_c : r = a'_c : r', \;\; b_a : r = b'_a : r'.$$

By aid of the distributive law, we obtain from these proportions the following:

$$a : r = a' : r, \;\; b : r = b' : r'.$$

Consequently, by virtue of the commutative law of multiplication, we have:

$$a : b = a' : b'$$

. $\square$

Further Hilbert writes: "From the theorem just demonstrated, we can easily deduce the fundamental theorem in the theory of proportion". At the same time, he does not provide either a drawing or a proof for Thales' theorem.

Theorem 2.1.7 (Thales by Hilbert)**.** *If two parallel lines cut from the sides of an arbitrary angle the segments a, b and a', b' respectively, then we always have the proportion $a : b = a' : b'$. Conversely, if the four segments a, b, a', b' fulfill this proportion and if a, a' and b, b' are laid off upon the two sides of an arbitrary angle respectively, then the straight lines joining the extremities of a and b and of a' and b' are parallel to each other.*

Let us prove Thales theorem in the Hilbert system.

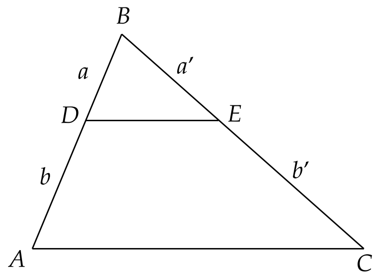


Figure 2.4: Proposition 2.1.6

Hartshorne completes of Hilbert. In his theory, he applies equivalence classes of congruence of line segments (Hartshorne 2000).

Definition addition of segments by Hartshorne:

Definition 2.1.8. Given congruence equivalence classes of line segments a, b, we define their sum as follows. Choose points A, B such that the segment AB represents the class a. Then on the line AB choose a point C with $A * B * C$[1], such that the segment BC represents the class b. Then we define $a + b$ to be represented by the segment AC.

Theorem 2.1.9. *In any Hilbert plane, addition of line segment classes has the following properties:*

(1) $a + b$ is well-defined, i.e., different choices of A, B, C in the definition will give rise to congruent segments.

(2) $a + b = b + a$, i.e., the corresponding line segments are congruent.

(3) $(a + b) + c = a + (b + c)$.

(4) Given any two classes a, b, one and only one of the following holds:

I) $a = b$.

[1]The notation A * B * C means that B is between A and C on some straight line

II) There is a class c such that $a + c = b$.

III) There is a class d such that $a = b + d$.

The above-mentioned properties can be proved on the basis of Hilbert axioms.

Hartshorne in his interpretation defines the multiplication of segments using an angle.

Definition 2.1.10. Given two segment classes a, b, we define their product ab as follows. First, make a right triangle ABC with $AB = 1$ and $BC = a$, where the right angle is at B. Let α be the angle $\angle BAC$. Second, make a new right triangle DEF with $DE = b$ and having the same angle α at D. Then we define ab to be the class of side EF of this new triangle (see Fig 2.5).

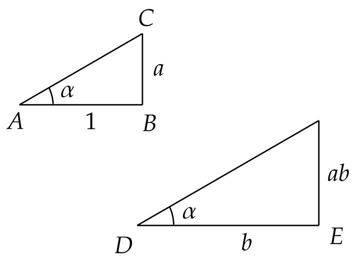


Figure 2.5: Definition of multiplication of segments by Hartshorne

The definition of multiplication of segments in the above manner allows Hartshorne to simplify the proof of the properties of multiplication.

Theorem 2.1.11. *In any Hilbert plane with parallel axiom, multiplication of segment classes has the following properties:*

(1) ab is well-defined.

(2) $a \cdot 1 = a$ for all a.

(3) $ab = ba$ for all a, b.

(4) $a(bc) = (ab)c$ for all a, b, c.

(5) For any a, there is a unique b such that $ab = 1$.

(6) $a(b + c) = ab + ac$ for all a, b, c.

Most of the theses from Proposition 2.1.11 Hartshorne proved using the property of cyclic quadrilaterals[2]: Let A, B, C, D be four points on the plane, with A, B both on the same side

[2] A cyclic quadrilateral is a set of four points A, B, C, D lying in that order on a circle, together with the lines AB, BC, CD, DA joining them. The lines AC and BD are the diagonals of the cyclic quadrilateral.

of the line CD. Then A, B, C, D lie on a circle if and only if the angles $\angle DAC$ and $\angle DBC$ are equal.

Hartshorne defines proportions in a slightly different way. First, he introduces the theorem: "Let P be a set, with two operations $+$, $\cdot$ defined on it that satisfy the properties listed in 2.1.11 and 2.1.9. Then there is a unique ordered field F whose positive elements form the set P.

He then further emphasizes, for any line segment AB, its congruence equivalence class a is an element of the field F. We will call a the length of AB. If AB and CD are two segments with lengths a, b, we can speak of their ratio as the quotient $\frac{a}{b} \in F$.

Definition 2.1.12. We say that four segments with lengths a, b, c, d are proportional if $\frac{a}{b} = \frac{c}{d}$ are elements of the field F.

Definition 2.1.13. Two triangles ABC and $A'B'C'$ are similar if the three angles of one are respectively equal to the three angles of the other, and the corresponding sides are proportional.

Theorem 2.1.14. *If two triangles ABC and DEF have their three angles respectively equal, then the two triangles are similar.*

The proof of this theorem is analogous to the proof of the theorem 2.1.6.

Theorem 2.1.15 (Thales by Hartshorne)**.** *In any triangle ABC, let $B'C'$ be drawn parallel to BC. Then the sides AB and AC are proportional to AB' and AC'. Conversely, if the sides are divided by points B', D such that AB, AC are proportional to AB', AD, then $B'D$ is parallel to BC.*

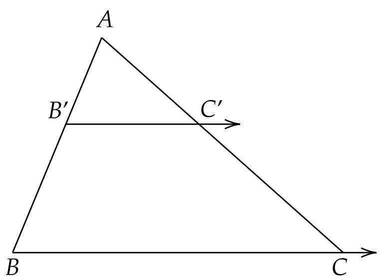


Figure 2.6: Thales by Hartshorne

Proof. Since $B'C'$ is parallel to BC, the angles at B', C' are equal to the angles at B, C, respectively. Since the angle at A is common, the triangles ABC and $AB'C'$ have their three angles equal, and so they are similar (Proposition 2.1.14). It follows that the sides are proportional.

Conversely, suppose we are given B', D such that AB, AC are proportional to AB', AD. Draw $B'C'$ parallel to BC. Then also AB, AC are proportional to AB', AC'. Since we are working in a field F, the fourth proportional to three given quantities is uniquely determined. Hence $AD \sim AC'$. Since the points D, C' lie on the same ray from A, the points D, C' are equal (axiom (C1)). Hence $B'D$ is parallel to BC. $\square$

Historically, Descartes was the first to introduce the arithmetic of line segments. It was based on Euclid's proportions, specifically on proposition VI.9; see (Błaszczyk, Mrówka 2015, Błaszczyk 2021). One can take Descartes' diagram as a definition (see Fig. 2.7). Modern systems that apply arithmetic of line segments seek to adjust Descartes' idea and requirements of axioms.

Definition 2.1.16. Let AB be taken as unity, and let it be required to multiply BD by BC. One has only to join the points A and C, and draw DE parallel to CA; then BE is the product of BD and BC.

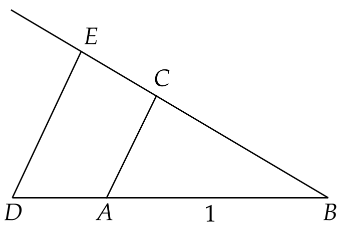


Figure 2.7: Descartes' definition of product

If it be required to divide BE by BD, One joins E and D, and draw AC parallel to DE; then BC is the result of the division.

2.2 Szmielew-Tarski's system

The way of arriving at Thales' theorem in this system is similar to Hilbert and Hartshorne: first introducing points arithmetic, then defining proportions (Schwabhäuser, Szmielew, Tarski

1983). The difference is that in the Szmielew-Tarski's system only points are the primitive objects. Before defining arithmetic operations over points, Szmielew-Tarski introduced the concept of axis. For the sake of simplicity, we can say that the axis is points o and e (or non-collinear points o, e, e'), where o is the beginning of the axis, e (and e') unit points.

Definition 2.2.1. Take in the plane three fixed points o, e and e', that define two fixed straight lines oe and oe'. Let e, a and b be colinear points. Now in order to define the sum of the points a and b, we construct $aa' \parallel ee'$ and draw through a' a parallel to oe and through b a parallel to oe'. Let these two parallels intersect in c^*. Finally, draw through c^* a straight line parallel to the ee'. Let this parallel cut the fixed line oe in c. Then c is called the sum of the a and b (see Fig. 2.8).

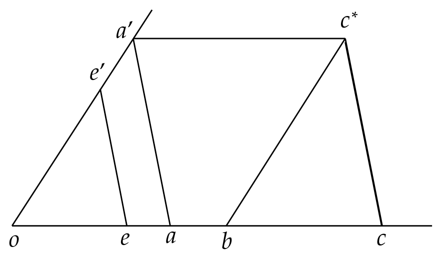


Figure 2.8: Definition of addition of points by Szmielew-Tarski

The definition of addition segments, based on the parallel projection, allows to have a definition which is correct for signed points.

Definition 2.2.2. Take in the plane three fixed points o, e and e' that define two fixed straight lines oe and oe'. Let e, a and b be colinear points. In order now to define the product of a and b determine upon oe' a point b' so that bb' is parallel to the unit-line ee', and join e' with a. Then draw through b' a straight line parallel to $e'a$. This parallel will intersect the fixed straight line oe in the point c, and we call c the product of the a by the b (see Fig. 2.9).

Using Pappus' and Desargues' theorems it can be shown, that for the such defined addition and product of points the associative, commutative and distributive law is valid. This, in turn, means that in the Szmielew-Tarski system, we must first prove the Poppus theorem, and on its basis, the Desargues theorem, which is not so easy. They also need proof of the proposition, that the addition and product of points are independent of the choice of point e' and point e.

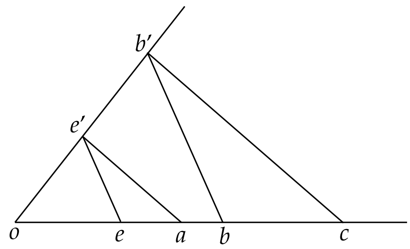


Figure 2.9: Definition of product of points by Szmielew-Tarski

The proofs of above-mentioned proposition are several dozen pages long, we do not repeat it here.

Next, authors defines positive points, such that lie on the ray which starts at o and the point e lies on it. Having positive elements, we can introduce the inequality $a \leq b$. We say that $a \leq b$ if $a = b$ or $a - b$ is positive. Based on positive points, we can define the difference, the relation $<$, $>$, $\leq$, $\geq$ and the square of the point.

All of this theory is the basis for the following definition:

Definition 2.2.3. We say that c is the length of the segment ab or is the distance between the points a, b with respect to axis oe, if c satisfies the conditions $o < c$ and $oc = ab$ and we denote $\overline{ab}$.

And only after all these reasoning and definitions, the theorem of Thales can be formulated.

Theorem 2.2.4 (Thales' theorem by Szmielew-Tarski)**.** *Let we have axis oe, collinear points p, a, b, collinear points p, c, d, collinear points p, a, c and $\overline{ac} \parallel \overline{bd}$ then $\overline{pa} \cdot \overline{pd} = \overline{pc} \cdot \overline{pb}$*

Authors noted, the theorem of 15.5 is of course just another way of expressing the "proportions" $\overline{pa} : \overline{pb} = \overline{pc} : \overline{pa}$ or $\overline{pa} : \overline{pc} = \overline{pb} : \overline{pd}$ (apart from the case $b = d = p$).

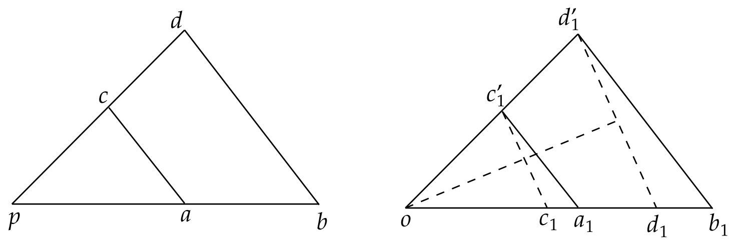


Figure 2.10: Proof of the Thales' theorem in Shmielew-Tarski system

Proof. Let $a_1 = \overline{pa}$, $b_1 = \overline{pb}$, $c_1 = \overline{pc}$, $d_1 = \overline{pd}$. We construct triangle c_1oc_1' congruent to triangle pac (see. Fig 2.10). Let m be midpoint of segment c_1c_1' and d_1' is point symetrical to point d_1 with respect to om. By properties of axial symmetry $od_1 = od_1' = pd$ and $om \perp d_1d_1'$. By the definition of right angle we have $om \perp c_1c_1'$. It means $c_1c_1' \parallel d_1d_1'$. By the condition, congruence SAS of triangles we have $\triangle pac = \triangle oa_1c_1'$ and $\triangle pbd = \triangle ob_1d_1'$. By the assumption $ac \parallel bd$. This means $\angle cap = \angle dbp$. Hence $\angle cap = \angle c_1'a_1o = \angle pbd = \angle ob_1d_1'$ and this means $c_1'a_1 \parallel d_1'b_1$. By the definition of product $a_1 \cdot d_1 = b_1 = c_1 \cdot b_1$. Hence $\overline{pa} \cdot \overline{pd} = \overline{pc} \cdot \overline{pb}$.

□

2.3 Borsuk-Szmielew's system

2.3.1 Measure in synthetic geometry

In (Borsuk, Szmielew 1972), an absolute (plane) geometry is the following system $(S, L, \mathbf{B}, \mathbf{D})$, where S is a set of points, L – a set of straight lines, $\mathbf{B}$ – a ternary relation of *betweenness* that holds among points, and $\mathbf{D}$ – a quaternary relation of *equidistance* that holds among points. Primitive concepts (point, straight line) and relations ($\mathbf{B}$, $\mathbf{D}$) are characterized by axioms of incidence, axioms regarding $\mathbf{B}$ and $\mathbf{D}$, and axioms regarding a congruence of line segments defined by $ab \equiv cd$ iff $\mathbf{D}(abcd)$. Finally, the system includes the continuity axiom characterizing an order of points on a straight line in terms of Dedekind cuts. In section ..., we spell out these axioms, herein we focus on how real numbers are introduced into that system. In short, it is related to a measure of (bound and free) line segments.

A theorem on existence and surjectivity of measure enables defining an isometry φ between a straight line L and the totally ordered set of real numbers $(\mathbb{R}, <)$, where $(\mathbb{R}, <)$ is considered a metric space, with the absolute value $| \ |$ being a metric. The measure φ, mapping the set of bounded line segments Z onto non-negative real numbers, is an intermediate step in defining φ.

First of all, the straight line gets a total order. Further definitions introduce concepts of bound line segments and triangles, enabling the formulation of axioms concerning congruences and the theorem on the existence of the midpoint of a segment (related to dyadic numbers). Then, on the set of segments, which we denote by Z, a linear order is defined; it has no algebraic structure, meaning one can not add or subtract these objects. The congruence of segments $\equiv$ is an equivalence relation on the set Z; a set of free segments, O, is introduced as a reduced product $O = Z/_{\equiv}$. It is endowed with a total order, addition, and multiplication by dyadic numbers. The set of dyadic numbers, which we denote by Θ, is countable and dense in the space $(\mathbb{R}_+, <)$. It appears crucial in the proof of the theorem on measure ψ on the set O: the existence of ψ draws on a relationship that each Dedekind cut in $(\Theta, <)$ determines exactly one (positive) real number, its surjectivity – on the observation that between two real numbers lies a dyadic number.

Theorem on the existence of measure (Borsuk, Szmielew 1960, 169). "Let a_0 be any free segment, and let x_0 be any real positive number. There exists just one measure ψ of free segments such that $\psi(a_0) = x_0$."

Given that $x_0 = 1$, the map ψ is defined as follows: Borsuk and Szmielew define a pair of subsets of dyadic numbers

$$\Theta_1^{(a)} = \{w \in \Theta_+ : wa_0 < a\}, \quad \Theta_2^{(a)} = \{w \in \Theta_+ : wa_0 \geq a\}.$$

It determines a Dedekind cut in $(\mathbb{R}, <)$, and consequently the unique real number, which we denote by x_a. The below formula defines map ψ,

$$\psi(a) = x_a.$$

The surjectivity of φ builds on the continuity axiom and the fact that the set Θ is dense in $(\mathbb{R}, <)$. To elaborate, let $a_0 \in L$ and b_0 be such that $b_0 > a_0$ and $\varphi(a_0 b_0) = x_0$. Then, firstly,

$$(\forall w \in \Theta)(\exists p \in L)[\varphi(a_0 p) = w x_0].$$

Secondly, given that $x \in \mathbb{R}_+$, the pair of subsets (X_1, X_2), where

$$X_1 = \{p \in L : \varphi(a_0 p) < x\}, \quad X_2 = \{p \in L : \varphi(a_0 q) \geq x\}$$

is a Dedekind cut in $(L, <)$. Due to the continuity axiom it determines a point on L, say, b, which proves to be such that $\varphi(a_0 b) = x$.

Below we tally structures, maps and relationships involved in the theorem on the existence measure:

(1) Bound segments make the totally ordered set $(Z, <)$.

(2) Free segments make a structure$(O, +, \circ, <)$, where map $\circ$ assigns to any dyadic number and any free segment another free segment, $\circ$: $\Theta \times O \mapsto O$, the relation $<$ is a total order, addition $+$ associative, commutative, and compatible with the order $<$, moreover the operation $\circ$ satisfies laws which we would rather ascribe Euclid's concept of magnitude, such as

(i) $w(a+b) = wa + wb$,

(ii) $(w+v)a = wa + va$,

(iii) $w(va) = (wv)a$,

(iv) $a < b \Rightarrow wa < wb$,

(v) $w < v \Rightarrow wa < va$,

(vi) $wa < wb \Rightarrow a < b$,

(vii) $wa < wb \Rightarrow w < v$,

(viii) $(\forall a < b)(\exists n \in \mathbb{N})[a < b \Rightarrow (na \leq b \wedge (n+1)a > b)]$,

(ix) $(\forall a, b)(\exists n \in \mathbb{N})[na > b)]$,

(x) $(\forall a, b)(\exists n \in \mathbb{N})[\frac{1}{2^k}a > b)]$,

(xi) $(\forall a, b, c)(\exists n, k \in \mathbb{N})[a < b \Rightarrow a < \frac{n+1}{2^k}c < b]$, where $w, v \in \Theta$, and $a, b \in O$.

(3) The straight line L is a linearly ordered set with a specific point a_0, $(L, a_0, <)$.

(4) The measure (the length) of bound segments $\varphi : Z \mapsto \mathbb{R}_+ \cup 0$.

(5) The measure (the length) of free segments $\psi : O \mapsto \mathbb{R}_+ \cup 0$.

(6) The relationship between maps φ and ψ is captured the following equality:

$$\psi([ab]) = \varphi(ab)$$

.

(7) An isometry $\Phi : L \ni p \mapsto x^p \in \mathbb{R}$, where

$$x^p = \begin{cases} \varphi(a_0p), \ \ \text{when} \ \ a_0 < p, \\ 0, \ \ \text{when} \ \ p = a_0, \\ -\varphi(a_0x), \ \ \text{when} \ \ p < a_0. \end{cases}$$

Map Φ also gets the name of coordinate on a line.

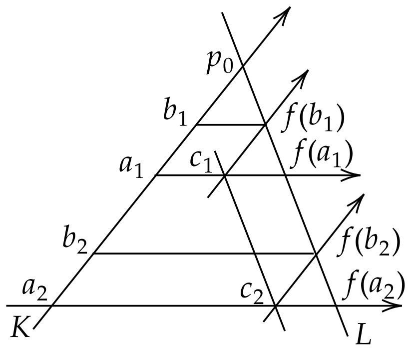
The below diagram summarizes this section.

$$\begin{tikzcd}
(Z, <) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathbb{R}_+ \cup \{0\} \\
(Z/_{\equiv}, +, \circ, <) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

2.3.2 Thales' theorem in terms of lengths of line segments

Thales Theorem is formulated with the help of the notions of parallel projectivity and similitude in the following way:

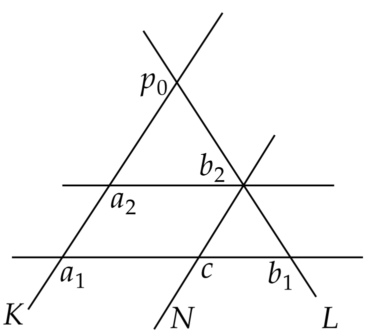
Any parallel projectivity f a line K upon a line L on a plane P is similitude.



As a consequence of Theorem authors obtain:

Consider in plane P two lines K, L intersecting in some point p_0 and two parallel lines M_1, M_2 not passing through point p_0 and intersecting line K in points a_1 and a_2, and line L in points b_1 and b_2 respectively. Then

$$\frac{\rho(p_0, a_1)}{\rho(p_0, a_2)} = \frac{\rho(p_0, b_1)}{\rho(p_0, b_2)} = \frac{\rho(a_1, b_1)}{\rho(a_2, b_2)}$$



2.4 Birkhoffs' system

Postulate I: Postulate of Line Measure. A set of points $\{A, B...\}$ on any line can be put into a $1:1$ correspondence with the real numbers $\{a, b...\}$ so that $|b-a| = d(A, B)$ for all points A and B.

Postulate II: Point-Line Postulate. There is one and only one line, l, that contains any two given distinct points P and Q.

Postulate III: Postulate of Angle Measure. A set of rays $\{l, m, n...\}$ through any point O can be put into $1:1$ correspondence with the real numbers $a(mod 2\pi)$ so that if A and B are points (not equal to O) of l and m, respectively, the difference $a_m - a_l(mod 2\pi)$ of the numbers associated with the lines l and m is angleAOB.

Postulate IV: Postulate of Similarity. Given two triangles ABC and $A'B'C'$ and some constant $k > 0$, $d(A', B') = kd(A, B)$, $d(A', C') = kd(A, C)$ and angle$B'A'C'$=$\pm$angleBAC, then $d(B', C') = kd(B, C)$, angle$C'B'A'$=$\pm$angleCBA, and angle$A'C'B'$=$\pm$angleACB (Birkhoff 1932).

In this system, the proof of Thales' theorem is obtained directly from the system of axioms.

2.5 Millman-Parker's system

Millman and Parker (Millman, Parker 1991) take point and line as a primitive concepts.

(i) For every two points A, B there is a line l with $A \in l$ and $B \in l$.

(ii) Every line has at least two points.

In the Millman-Parker system, there are no primitive relations, but there is a metric d (distance between points).

Definition 2.5.1. A distance function on set of points G is a function $d : G \times G \to \mathbb{R}$ such that for all $P, Q \in G$.

(i) $d(P, Q) \geq 0$;

(ii) $d(P, Q) = 0$ if and only if $P = Q$;

(iii) $d(P, Q) = d(Q, P)$.

The relationship "lying between" is defined by a metric. $A - B - C$ means that B lies between A and C, AB means the distance (A, B), the distance is a real number. In this notation, $A - B - C$ holds if and only if $AB + BC = AC$.

Definition 2.5.2. Let l be a line. Assume that there is a distance function d. A function $f : l \to \mathbb{R}$ is ruler (or coordinate system) for l if

(i) f is a bijection;

(ii) for each pair of points P and Q on l

$$|f(P) - f(Q)| = d(P, Q).$$

In Milman-Parker's geometry, real number arithmetic is related to metric, but the proof of Thales' theorem is also difficult. Additional theorems are needed to make this proof.

Theorem 2.5.3. *In metric geometry, any segment can be divided into n congruent parts for any $n > 0$.*

In the proof, we use the statement that there is a bijection between the straight line and the real numbers. Each point on this straight line can be the start of the coordinate system, the beginning of the segment will be it too. Then if we divide the length of the segment d into n, we get the real number $\frac{d}{n}$, which always has a preimage on the line. Putting aside the segments of $\frac{d}{n}$ in turn, we will get the points we are looking for.

One more necessary theorem to the proof Thales' theorem is the theorem about "lying between".

Theorem 2.5.4. *Let l_1, l_2 and l_3 be distinct parallel lines. Let t_1 intersect l_1, l_2 and l_3 at A, B, and C (respectively) and let t_2 intersect l_1, l_2 and l_3 at D, E and F (respectively). If $A - B - C$ then $D - E - F$.*

Next is a formulated theorem that resembles Thales' theorem for equal segments.

Theorem 2.5.5. *Let l_1, l_2, l_3 be distinct parallel lines. Let t_1 intersect l_1, l_2, l_3 at A, B, C (respectively) and let t_2 intersect l_1, l_2, l_3 at D, E i F (respectively). If $\overline{AB} \simeq \overline{BC}$ then $\overline{DE} \simeq \overline{EF}$ ($\overline{AB}$ notation means the segment AB).*

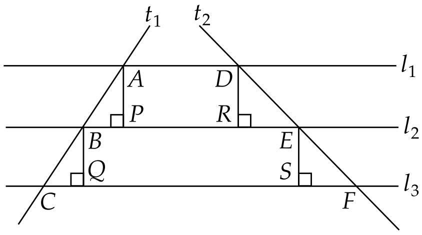


Figure 2.11: Theorem 2.5.5

Proof. The proof of this theorem is geometric (see Fig. 2.11).

Since $AB = BC$ and $l_1 \neq l_3$, then $A \neq C$ and $A - B - C$. Let P be the foot of the perpendicular from A to l_2, Q be the foot of the perpendicular from B to l_3, R the foot of the perpendicular from D to l_2, and S the foot of the perpendicular from E to l_3. Then according to the rule of SAA $\triangle ABP \simeq \triangle BCQ$ ($AB = BC$, $\angle ABP \simeq \angle BCQ$), it means $\overline{AP} \simeq \overline{BQ}$. But also $\overline{AP} \simeq \overline{DR}$ and $\overline{ES} \simeq \overline{BQ}$. Hence $\overline{DR} \simeq \overline{ES}$. Consequently by the congruence of triangles DRE and ESF we get $\overline{DE} \simeq \overline{EF}$.

□

Theorem 2.5.6 (Thales' theorem by Millman-Parker)**.** *Niech l_1, l_2 i l_3 będą rożnymi prostymi równoległymi. Niech t_1 i t_2 będą dwiema poprzecznymi, które przecinają l_1, l_2 i l_3 w punktach A, B, C i D, E, F tak, ze $A - B - C$, jak na rysunku 2.12. Wtedy:*

$$\frac{BC}{AB} = \frac{EF}{DE}$$

Note: Since the lengths are real numbers, and in the field of real numbers the arithmetic operations $+, -, \cdot, :$ are defined, the notation $\frac{BC}{AB} = \frac{EF}{DE}$ means the real number operations.

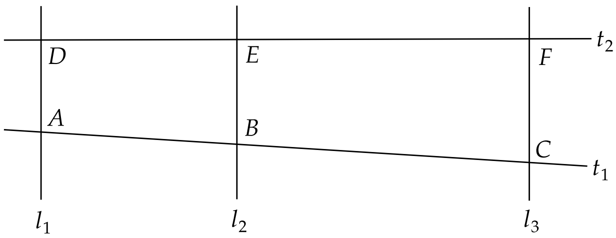


Figure 2.12: Theorem 2.5.5

Proof. For the proof, it is enough to show that the following inequality holds:

$$(\forall n \in \mathbb{N}) \left| \frac{BC}{AB} - \frac{EF}{DE} \right| < \frac{1}{n}.$$

If above inequality holds for all $n > 0$, especially for n very large, $\left|\frac{BC}{AB} - \frac{EF}{DE}\right|$ must be zero, which proves equation $\frac{BC}{AB} = \frac{EF}{DE}$.

Note: We can see that in the proof the authors directly refer to the properties of the limits in the field of real numbers.

(1) Let's choose any n. Let p be the largest non-negative integer such that $\frac{p}{n} \leq \frac{BC}{AB}$. Such a number of course exists, namely $p = \max\left\{k \in \mathbb{N} : k \leq \frac{nBC}{AB}\right\}$.

So we have:

$$\frac{p}{n} \leq \frac{BC}{AB} \leq \frac{p+1}{n}. \tag{2.1}$$

We shall break the segment $\overline{AB}$ into n segments each of length $\frac{AB}{n}$ (tw. 2.5.3) and then lay off $p+1$ segments of this same length along $\overline{BC}$.

Let $A_1, ..., A_{q-1}$ be points of $\overline{AB}$ with $A - A_1 - \cdots - A_{q-1} - B$ and each segment $\overline{A_i A_{i+1}}$ has length $\frac{AB}{n}$.

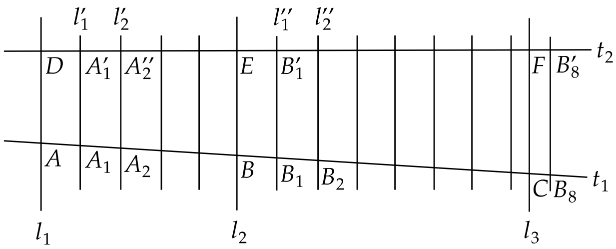


Figure 2.13: Illustration for the case $n = 5$ i $p = 7$

Similary choose points $B_1, B_2, ..., B_{p+1}$ on $\overline{BC}$ so that the distance between neighboring points is $\frac{AB}{n}$ (see Fig. 2.13).

(2) Show that $C \in B_p B_{p+1}$. By inequality (2.1) we have $BB_p = p \cdot BB_1 = \frac{p}{n} \cdot AB \leq BC$. This means, that $B_p \in \overline{BC}$. Similarly $B_{p+1} \notin BC$. Thus $B - C - B_{p+1}$. This means either $C = B_p$ or $B_p - C - B_{p+1}$. Hence $C \in B_p B_{p+1}$ i $C \neq B_{p+1}$.

(3) Let l'_i be the line through A_i parallel to $l_1 = \overleftrightarrow{AD}$ and l_i intersects t_2 at a point A'_i. Similarly let l''_j be the line through B_j parallel to l_1. This line intersects t_2 at a point B'_j. By

theorems 2.5.4 and 2.5.5 we have $\overline{DE}$ is breaked into n segments each of length $\frac{DE}{n}$, and along $\overline{EB}$ we have set aside $p+$ 1 segments of the same length.

Similarly as in parts (1) and (2) we show that $p \cdot EB'_1 = EB'_p \leq EF < EB'_{p+1} = (p+1) \cdot EB'_1$. From the equality of the segments we get $p \cdot DA'_1 \leq EF < (p+1) \cdot DA'_1$.

Dividing this inequality by DE and remembering that $DE = n \cdot DA'_1$ we get:

$$\frac{p}{n} \leq \frac{EF}{DE} < \frac{p+1}{n},$$

otherwise:

$$-\frac{p+1}{n} < -\frac{EF}{DE} \leq -\frac{p}{n},$$

Adding this to the inequality

$$\frac{p}{n} \leq \frac{BC}{AB} \leq \frac{p+1}{n}$$

we have

$$-\frac{1}{n} < \frac{BC}{AB} - \frac{EF}{DE} < \frac{1}{n}.$$

That is

$$\left| \frac{BC}{AB} - \frac{EF}{DE} \right| < \frac{1}{n}.$$

$\square$

Chapter 3

The Area Method and Thales's Theorem

3.1 Axioms of the area method

The area method, pioneered in (Chou, Gao, Zhang 1994), is a technique of proving theorems and constructing solutions in Euclidean geometry. (Janičić, Narboux, Quaresma 2012) provides its axiomatic description. In this section, we introduce that system. In (Błaszczyk, Petiurenko 2019), we present a model for these axioms.

From the perspective of formal systems, the language of the area method includes one kind of variables, and symbols of a binary, $\overline{}$, and a ternary function, S. We also need the language of an ordered field, that is, symbols of binary functions, $+$, $\cdot$ (sum and product), and unary functions $-,^{-1}$ (an opposite and inverse element), as well as constants $0, 1$, and finitely many constants and r. Less formally, there are three primitive notions in the area method: point, length of a directed segment, and a signed area of a triangle. An ordered pair of points is called a directed segment, an ordered triple – a triangle. In what follows, capital letters A, B, C, etc., stand for points. The length of a directed segment, $\overline{AB}$, in short, is an element of an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$.[1] Similarly, the signed area of a triangle, S_{ABC}, in short, is an element of the ordered field. $\overline{AB}$ and S_{ABC} can be positive, negative, or zero and they are processed in the arithmetic of an ordered field.

To model Euclidean geometry, we need some definitions that we apply in axioms.[2]

[1] Adopting a formal perspective, it can be a commutative field of characteristics other than 2. Yet standard models of elementary geometry are Cartesian planes over an ordered field.

[2] (Hartshorne 2000) relates these definitions with Euclid's original system

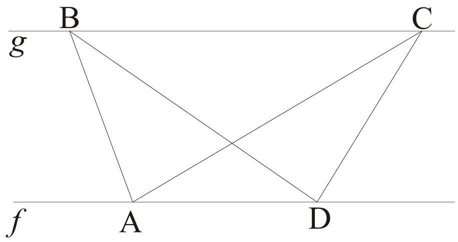


Figure 3.1: Definition of parallel line segments

Definition 3.1.1. Points A, B, C are collinear iff $S_{ABC} = 0$.

Definition 3.1.2. Two segments AD and BC, where $A \neq D$ and $B \neq C$, are parallel, iff $S_{ABC} = S_{DBC}$. For this relation, we adopt the standard symbol $AD \parallel BC$.

Definition 3.1.3. For three points A, B and C, the Pythagorean difference, denoted by P_{ABC}, is defined by

$$P_{ABC} = \overline{AB}^2 + \overline{BC}^2 - \overline{AC}^2.$$

Definition 3.1.4. Two segments DB and CA, where $D \neq B$ and $C \neq A$, are perpendicular iff $P_{DCA} = P_{BCA}$. This relation is denoted by $DB \perp CA$.

Here is the first proposition of the *Elements*, Book VI.

Theorem 3.1.5 (*Elements*, VI.1)**.** *In triangles, ABC and ACD, having the same height AC, as base BC is to base CD, so triangle ABC is to triangle ACD.*

In school geometry, where the product of base and height determines the area of a triangle, that theorem follows from algebraic identity $\frac{a_1 h}{a_2 h} = \frac{a_1}{a_2}$. However, Euclid's proof proceeds within proportion theory; it involves the non-defined addition of triangles and requires comparing triangles as greater–lesser (Błaszczyk 2018). Both concepts seem foreign to a modern reader. In this paper, we adopt VI.1 as the axiom A10. Thus, on the one hand, we do not need to refer to Euclid's definition of proportion. On the other, we do not need to apply any formula for an area of a triangle. Specifically, our account does not involve products of line segments or their measures (i.e., real numbers). Moreover, although objects $\overline{AB}$ and S_{ABC} are elements of an ordered field, we process only terms such as

$$\frac{\overline{AB}}{\overline{CD}} = \frac{\overline{EF}}{\overline{GH}}, \quad \frac{\overline{AB}}{\overline{CD}} = \frac{S_{EFG}}{S_{GIJ}}.$$

To this end, we apply Euclid's propositions regarding proportions, namely V.12, V.17–19, that are also laws of fractions in an ordered field. In an algebraic stylization, these propositions are as follows:[3]

V.12 $\frac{a}{b} = \frac{c}{ref_R H}, \ \frac{a}{b} = \frac{e}{f} \Rightarrow \frac{a}{b} = \frac{(a+c+e)}{(b+d+f)}$.

V.17 $\frac{(a+b)}{b} = \frac{(c+d)}{d} \Rightarrow \frac{a}{b} = \frac{c}{d}$.

V.18 $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{(a+b)}{b} = \frac{(c+d)}{ref_R H}$.

V.19 $\frac{(a+b)}{(c+d)} = \frac{a}{c} \Rightarrow \frac{b}{d} = \frac{(a+b)}{(c+d)}$.

In Euclid's theory, $a, b, c, ...$ stand for the so-called magnitudes, that is, line segments, triangles, figures and solids, and angles. In the area method, we apply propositions V.12, 17–19 to directed line segments and signed areas.

Combining V.12 and V.19 we get

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{a+c}{b+d}.$$

We will refer to Euclid's propositions through that version. Furthermore, we will apply V.19 in the following form

$$\frac{(a+b)}{(c+d)} = \frac{a}{c} \Rightarrow \frac{a}{c} = \frac{b}{ref_R H}.$$

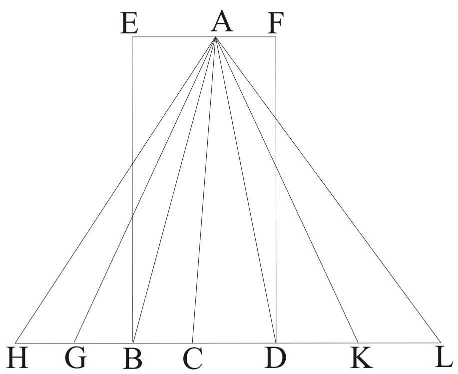


Figure 3.2: *Elements*, VI.1

Here are the axioms for the area method (Janičić, Narboux, Quaresma 2012).

[3]For a discussion of reconstruction of Euclid's Book V in an ordered field, see (Błaszczyk, Petiurenko 2019).

A1. $\overline{AB} = 0$ if and only if A and B are identical.

A2. $S_{ABC} = S_{CAB}$.

A3. $S_{ABC} = -S_{BAC}$.

A4. If $S_{ABC} = 0$, then $\overline{AB} + \overline{BC} = \overline{AC}$ (Chasles' axiom).

A5. There are points A, B and C such that $S_{ABC} \neq 0$ (not all points are collinear).

A6. $S_{ABC} = S_{DBC} + S_{ADC} + S_{ABD}$ (all points are in the same plane).[4]

A7. For each element r of F, there exists a point P, such that $S_{ABP} = 0$ and $\overline{AP} = r\overline{AB}$ (construction of a point on a line).

A8. If $A \neq B$, $S_{ABP} = 0, \overline{AP} = r\overline{AB}$, $S_{ABP'} = 0$ and $\overline{AP'} = r\overline{AB}$, then $P = P'$.

A9. If $PQ \parallel CD$ and $\frac{\overline{PQ}}{\overline{CD}} = 1$, then $DQ \parallel PC$ (parallelogram).

A10. If $S_{PAC} \neq 0$ and $S_{ABC} = 0$, then $\frac{\overline{AB}}{\overline{AC}} = \frac{S_{PAB}}{S_{PAC}}$ (Euclid's proposition VI.1).

A11. If $C \neq D$ and $AB \perp CD$ and $EF \perp CD$, then $AB \parallel EF$.

A12. If $A \neq B$, $AB \perp CD$ and $AB \parallel EF$, then $EF \perp CD$.

A13. If $FA \perp BC$ and $S_{FBC} = 0$, then $4 \cdot S_{ABC}^2 = \overline{AF}^2\overline{BC}^2$ (formula for the area of a triangle).

3.2 Thales's Theorem in Euclid's Elements and in the Area Method

We present Euclid's proposition VI.2, the so-called Thales' theorem. The English translation reads:

Theorem 3.2.1. *If some straight line is drawn parallel to one of the sides of a triangle then it will cut the sides of the triangle proportionally. And if the sides of a triangle are cut proportionally then the straight line joining the cutting will be parallel to the remaining side of the triangle.*

First, we present Euclid's proof in a schematized form. To this end, we apply standard symbols such as $||$ or $\perp$ meaning parallel or perpendicular lines respectively. Second, we use

[4]The idea of signed area originates from Hilbert's *Foundations of Geometry.* In (Hilbert 1972, ch. 5) he proves the theorem that is a counterpart of axiom A6.

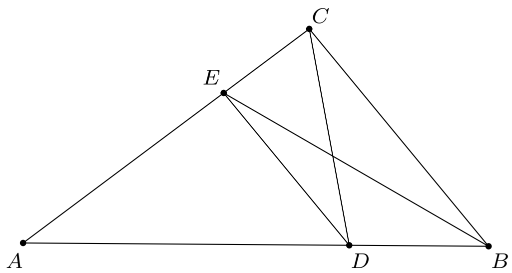


Figure 3.3: Thales's Theorem

specific symbols such as $\xrightarrow[I.38]{}$. Here, the arrow represents connective "for" rather than the logical implication, and the subscript I.38 means that there is a reference to proposition I.38; in the *Elements*, references are rendered by some characteristic phrases. Thus, the first two sentences of Euclid's proof: "Thus, triangle BDE is equal to triangle CDE. For they are on the same base DE and between the same parallels DE and BC", we represent by the following scheme:

$$DE \| BC \xrightarrow[I.38]{} \triangle(BDE) = \triangle(CDE)$$

In fact, to simplify our presentation, we skip the assumption that triangles BDE and CDE are "on the same base DE".

Euclid's second argument is covered by three sentences: "And ADE is some other triangle. And equal things have the same ratio? Thus, as triangle BDE is to ADE, so triangle CDE (is) to triangle ADE". Here, the second sentence literally cites the thesis of proposition V.7. Thus, in accordance with the conventions we adopted, we turn this argument into the following scheme

$$\triangle(ADE) \xrightarrow[V.7]{} \triangle(BDE) : \triangle(ADE) :: \triangle(CDE) : \triangle(ADE).$$

Euclid's next argument is this: "But, as triangle BDE (is) to triangle ADE, so (is) BD to DA. For, having the same height –(namely), the (straight line) drawn from E perpendicular to AB – they are to one another as their bases". We represent it by this scheme

$$E \perp AB \xrightarrow[VI.1]{} \triangle(BDE) : \triangle(ADE) :: BD : DA,$$

where $E \perp AB$ represents the the stipulation "having the same height –(namely), the (straight-line) drawn from E perpendicular to AB".

Now, the first part of Euclid's proof we represent as follows:

$$\begin{array}{rcl}
DE \| BC & \xrightarrow[I.38]{} & \triangle(BDE) = \triangle(CDE) \\
\triangle(ADE) & \xrightarrow[V.7]{} & \triangle(BDE) : \triangle(ADE) :: \triangle(CDE) : \triangle(ADE) \\
E \perp AB & \xrightarrow[VI.1]{} & \triangle(BDE) : \triangle(ADE) :: BD : DA \\
 & \xrightarrow[\delta]{} & \triangle(CDE) : \triangle(ADE) :: CE : EA \\
 & \xrightarrow[V.11]{} & BD : DA :: CE : EA.
\end{array}$$

Letter δ represents here the phrase διά τά ἀυτά (for the same reason).

The second part of Euclid's proof, that is, from proportionality to parallel lines, is as follows

$$\begin{array}{rcl}
BD : DA :: CE : EA, & & \\
BD : DA :: \triangle(BDE) : \triangle(ADE), & & \\
CE : EA :: \triangle(CDE) : \triangle(ADE) & \xrightarrow[V.11]{} & \triangle(BDE) : \triangle(ADE) :: \\
 & & :: \triangle(CDE) : \triangle(ADE) \\
\triangle(BDE) : \triangle(ADE) :: & & \\
:: \triangle(CDE) : \triangle(ADE) & \xrightarrow[V.9]{} & \triangle(BDE) = \triangle(CDE), DE \\
 & \xrightarrow[I.39]{} & DE \| BC.
\end{array}$$

Now we present a proof within the area method. To underline the analogy, we number steps in this proof.

[1] From the assumption $DE \parallel BC$, by definition we get the equality of signed areas $S_{DEB} = S_{DEC}$.

[2] By the rules of an ordered field,

$$\frac{S_{DEB}}{S_{DAE}} = \frac{S_{DEC}}{S_{DAE}}.$$

[3] By A2, we can permute names of vertices, then by A10, the following qualities obtain the following qualities:

$$\frac{S_{BDE}}{S_{DAE}} = \frac{\overline{BD}}{\overline{DA}}, \quad \frac{S_{CDE}}{S_{DAE}} = \frac{\overline{CE}}{\overline{EA}}.$$

[4] By transitivity of equality in an ordered field,

$$\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}.$$

For the proof of the second part, suppose $\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}$.

[1] By A10:

$$\frac{S_{BDE}}{S_{DAE}} = \frac{\overline{BD}}{\overline{DA}}, \; \frac{S_{CED}}{S_{AED}} = \frac{\overline{CE}}{\overline{AE}}.$$

[2] By transitivity of equality:

$$\frac{S_{BDE}}{S_{DAE}} = \frac{S_{CED}}{S_{AED}}.$$

[3] By axiom A2, $S_{DAE} = S_{AED}$. Applying arithmetic of an ordered field to the above equality we have

$$S_{BDE} = S_{CED}.$$

[4] By definition of parallel lines, $DE \parallel BC$ □

We skip the proof of the second part.

The schemes of Euclid's proof and the area method proof are almost identical. The only difference is, within the area method, we must respect the order of end-points of line segments and vertices of triangles.

3.3 Model of the area method

3.3.1 Interpreting primitive notions

In this section, we provide a model for axioms A1–A13. We interpret the set of points as elements of the Cartesian plane $\mathbb{R} \times \mathbb{R}$. Thus, a point A is an ordered pair of real numbers. In this chapter, we assume that $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3),\ D = (x_4, y_4)$.

The lexicographical order in the plane $\mathbb{R} \times \mathbb{R}$ is defined as follows:

Definition 3.3.1.

$$A \preceq B \iff x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 \leqslant y_2)\,,$$

where $x_1 < x_2$ is the inequality of real numbers.

Definition 3.3.2. The length of a directed segment AB is the number $\overline{AB}$ defined by

$$\overline{AB} = \left\{ \begin{array}{ll} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, & \text{when } A \preceq B, \\ -\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, & \text{when } B \preceq A. \end{array} \right\}$$

Definition 3.3.3. The signed area for a triangle ABC is the number S_{ABC} defined by

$$S_{ABC} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note that in the analytic geometry, the number $|S_{ABC}|$ stands for the area of the triangle with vertices A, B, C. In our interpretation, the absolute value is omitted. Hence, the number S_{ABC} can be positive or negative.

3.3.2 Co-linearity, parallelism, perpendicularity

We show that in our model, Definitions 1-4 introduce the standard meaning of (a) collinearity of points, as well as (b) parallel and (c) perpendicular lines.

Regarding (a), first, we show that if points A, B, C are collinear, then $S_{ABC} = 0$. Let then A, B, C lie on a line given by the equation $y = ax + b$. Then we have

$$y_1 = ax_1 + b,\ y_2 = ax_2 + b,\ y_3 = ax_3 + b.$$

Hence,

$$S_{ABC} = \frac{1}{2} \begin{vmatrix} x_1 & ax_1 + b & 1 \\ x_2 & ax_2 + b & 1 \\ x_3 & ax_3 + b & 1 \end{vmatrix} = 0.$$

Suppose now that $S_{ABC} = 0$ and A, B lie on the line $y = ax + b$. It means, the equalities hold

$$y_1 = ax_1 + b,\ y_2 = ax_2 + b.$$

From

$$S_{ABC} = \begin{vmatrix} x_1 & ax_1 + b & 1 \\ x_2 & ax_2 + b & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

it follows that

$$x_1(ax_2 + b) + x_2y_3 + x_3(ax_1 + b) - x_3(ax_2 + b) - x_1y_3 - x_2(ax_1 + b) = 0.$$

As a result, we get $y_3 = ax_3 + b$. .

Regarding (b), let

$$S_{ACD} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}, \; S_{BCD} = \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}.$$

If $S_{ACD} = S_{BCD}$, then

$$\frac{y_1 - y_2}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}.$$

This means that the slopes of the lines on which points A, B, on the one hand, and C, D, on the other, lie, are equal. As a result, lines running through points A, B and C, D respectively, are parallel.

Finally, suppose A, B and C, D lie on parallel lines, that is

$$y_1 = ax_1 + b, \;\; y_2 = ax_2 + b \;\; \text{and} \;\; y_3 = ax_3 + c, \;\; y_4 = ax_4 + c.$$

Hence,

$$S_{ACD} = \frac{1}{2}\begin{vmatrix} x_1 & ax_1 + b & 1 \\ x_3 & ax_3 + c & 1 \\ x_4 & ax_4 + c & 1 \end{vmatrix} = \frac{1}{2}(x_3c + x_4b - x_4c - x_3b).$$

$$S_{BCD} = \frac{1}{2}\begin{vmatrix} x_2 & ax_2 + b & 1 \\ x_3 & ax_3 + c & 1 \\ x_4 & ax_4 + c & 1 \end{vmatrix} = \frac{1}{2}(x_3c + x_4b - x_4c - x_3b).$$

This means

$$S_{ACD} = S_{BCD}.$$

We can conclude that in our model, the relationship $\|$ as given by Definition 2 is the same as parallelity in analytic geometry.

Regarding (c), suppose segments DB and CA are perpendicular in the sense of Definition 2, i.e., $P_{DCA} = P_{BCA}$

From the definition of perpendicularity we get:

$$P_{DCA} = \overline{DC}^2 + \overline{CA}^2 - \overline{DA}^2$$

$$P_{BCA} = \overline{BC}^2 + \overline{CA}^2 - \overline{BA}^2$$

$$\overline{DC}^2 - \overline{DA}^2 = \overline{BC}^2 - \overline{BA}^2$$

$$\frac{y_4 - y_2}{x_1 - x_3} = \frac{x_4 - x_2}{y_1 - y_3}$$

This means that the slopes of the lines AB and CD are in inverse proportion to each other and have opposite signs. Thus, we can conclude that the relationship $\perp$ in the defined model means perpendicularity in analytical geometry.

Suppose D, B and C, A lie on perpendicular lines, that is, the product of $\overline{DB}$ and $\overline{CA}$ is zero.

$$(\overline{DB} \cdot \overline{CA}) = (x_2 - x_4)(x_1 - x_3) + (y_2 - y_4)(y_1 - y_3) = 0$$

$$x_2x_1 - x_2x_3 - x_4x_1 + x_4x_3 + y_2y_1 - y_2y_3 - y_4y_1 + y_4y_3 = 0$$

$$P_{DCA} = 2{x_3}^2 - 2x_3x_4 - 2x_1x_3 + 2x_1x_4 + 2{y_3}^2 - 2y_3y_4 - 2y_1y_3 + 2y_1y_4$$

$$P_{BCA} = 2{x_3}^2 - 2x_3x_2 - 2x_1x_3 + 2x_1x_2 + 2{y_3}^2 - 2y_3y_2 - 2y_1y_3 + 2y_1y_2$$

$$P_{DCA} = P_{BCA} \iff P_{DCA} - P_{BCA} = 0$$

$$P_{DCA} - P_{BCA} = -2(x_2x_1 - x_2x_3 - x_4x_1 + x_4x_3 + y_2y_1 - y_2y_3 - y_4y_1 + y_4y_3) = 0$$

3.3.3 Interpreting axioms

In this section, we show that under the interpretation of section 2.1, all axioms of the area method are theorems of the analytic geometry on the Cartesian plane $\mathbb{R} \times \mathbb{R}$ with lexicographic order. Since parallelism and perpendicularity in our interpretation mean the same as parallelism and perpendicularity in analytic geometry, then axioms A9, A11, A12, A13 are well-known theorems.

Ad A1. Given $A = (x_1, y_1)$ and $B = (x_2, y_2)$, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0$ if and only if $x_1 = x_2$, $y_1 = y_2$.

A2 and A3 follow from properties of determinants.

Ad A2. Indeed, the equality $S_{ABC} = S_{CAB}$ is obtained, because

$$S_{CAB} = \frac{1}{2}\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = S_{ABC}.$$

Ad A3. In a similar way, the equality $S_{ABC} = -S_{BAC}$ is the result of the interchanging of two rows of the determinant rule.

Axiom 3.3.4. If $S_{ABC} = 0$, then $\overline{AB} + \overline{BC} = \overline{AC}$

$S_{ABC} = 0 \Rightarrow$ points A, B, C are collinear. Let's consider different cases of the location of these points according to lexicographic order. Let, $C \preceq A \preceq B$.

$$\overline{AB} = |AB|,\ \overline{BC} = -|BC|,\ \overline{AC} = -|AC|,$$

$$|AB| - |BC| = -|AC| \Rightarrow \overline{AB} + \overline{BC} = \overline{AC}$$

It can be shown analogously that dependence is met for all possible cases.

Axiom 3.3.5. There are points A, B i C, such that $S_{ABC} \neq 0$

Let's show an example of points for which the signed area is not equal to 0.

$$S_{BAC} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}$$

Axiom 3.3.6. $\mathcal{S}_{ABC} = \mathcal{S}_{DBC} + \mathcal{S}_{ADC} + \mathcal{S}_{ABD}$ (dimensionally, all points are on the same plane)

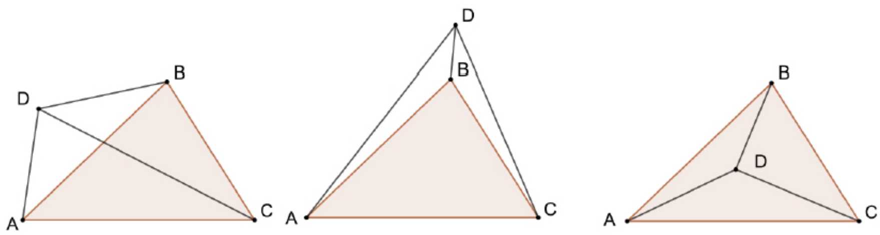


Figure 3.4: $\mathcal{S}_{ABC} = \mathcal{S}_{DBC} + \mathcal{S}_{ADC} + \mathcal{S}_{ABD}$

$$S_{ABC} = \frac{1}{2}(x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_1y_3 - x_2y_1).$$

$$S_{DBC} = \frac{1}{2}(x_4y_2 + x_3y_4 + x_2y_3 - x_3y_2 - x_4y_3 - x_2y_4).$$

$$S_{ADC} = \frac{1}{2}(x_1y_4 + x_3y_1 + x_4y_3 - x_3y_4 - x_1y_3 - x_4y_1).$$

$$S_{ABD} = \frac{1}{2}(x_1y_2 + x_4y_1 + x_2y_4 - x_4y_2 - x_1y_4 - x_2y_1).$$

$$\mathcal{S}_{DBC} + \mathcal{S}_{ADC} + \mathcal{S}_{ABD} = \mathcal{S}_{ABC}$$

Axiom 3.3.7. For each element r of F, there exists a point P, such that $S_{ABP} = 0$ and $\overline{AP} = r\overline{AB}$ (construction of a point on the line).

Let $A = (x_1, y_1), B = (x_2, y_2), P = (x, y)$. $S_{ABP} = 0 \Rightarrow A, B, P$ - are collinear (are on line $y = ax + b$)

$$\pm\sqrt{(x - x_1)^2 + (y - y_1)^2} = \pm r\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$|x - x_1|\sqrt{a^2 + 1} = r|x_2 - x_1|\sqrt{a^2 + 1}$$
$$|x - x_1| = r|x_2 - x_1| \Rightarrow r = \left|\frac{x - x_1}{x_2 - x_1}\right|$$

Based on the last equation, we can calculate the x coordinate of the point P with given A, B, r. We will calculate the y coordinate from the equation of the line $y = ax + b$, through the points A and B.

Axiom 3.3.8. If $A \neq B$, $\mathcal{S}_{ABP} = 0$, $\overline{AP} = r\overline{AB}$, $\mathcal{S}_{ABP'} = 0$ and $\overline{AP'} = r\overline{AB}$, then $P = P'$ (unicity)

$S_{ABP} = 0 \Rightarrow A, P, P'$ - are collinear. $\overline{AP} = r\overline{AB} \wedge \overline{AP'} = r\overline{AB} \Rightarrow P = P'$

Axiom 3.3.9. If $\mathcal{S}_{PAC} \neq 0$ and $\mathcal{S}_{ABC} = 0$, then $\frac{\overline{AB}}{\overline{AC}} = \frac{\mathcal{S}_{PAB}}{\mathcal{S}_{PAC}}$

$S_{ABC} = 0 \rightarrow A, B, C$ - are collinear and are on line $y = ax + b$. $\mathcal{S}_{PAC} \neq 0 \rightarrow P, A, C$ are not collinear. Let $A = (x_1, ax_1 + b), B = (x_2, ax_2 + b), C = (x_2, ax_2 + b), P = (x_4, y_4)$

$$\overline{AB} = \pm\sqrt{(x_2 - x_1)^2 + (ax_2 - ax_1)^2} = |x_2 - x_1|\sqrt{a^2 - 1}$$

$$\overline{AC} = \pm\sqrt{(x_3 - x_1)^2 + (ax_3 - ax_1)^2} = |x_x - x_1|\sqrt{a^2 - 1}$$

$$\frac{\overline{AB}}{\overline{AC}} = \frac{x_2 - x_1}{x_3 - x_1}$$

$$S_{PAB} = \frac{1}{2}(x_1 - x_2)(ax_4 - y_4 + b)$$

$$S_{PAC} = \frac{1}{2}(x_1 - x_3)(ax_4 - y_4 + b)$$

$$\frac{S_{PAB}}{S_{PAC}} = \frac{(x_1 - x_2)(ax_4 - y_4 + b)}{(x_1 - x_3)(ax_4 - y_4 + b)} = \frac{(x_2 - x_1)}{(x_3 - x_1)} = \frac{\overline{AB}}{\overline{AC}}$$

We have shown that all axioms of field theory [] are theorems in geometry on the Cartesian plane with lexicographic order, so any theorem that can be proved by the area methods is also a theorem in analytical geometry.

3.4 Co-side Theorem

3.4.1 Proof of Co-side Theorem

The so-called Co-side Theorem is the fundamental tool of the area method.

Theorem 3.4.1. *For four distinct points A, B, P, Q, let M be the intersection of the lines AB and PQ such that $Q \neq M$. Then the following equality is obtained*

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

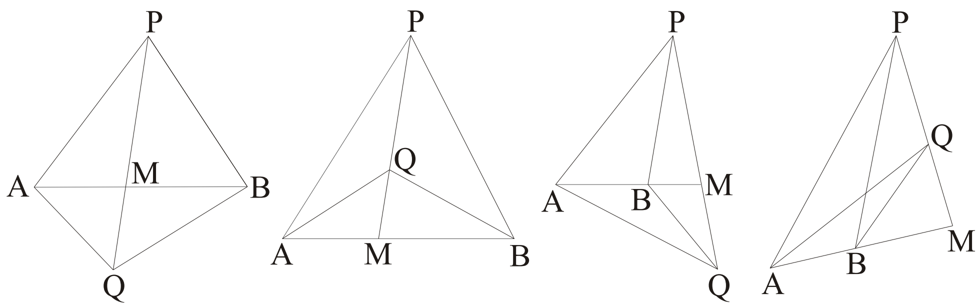


Figure 3.5: Co-side Theorem

Below we present two proofs. The first comes from ((Chou, Gao, Zhang 1994), pp. 8–17). It builds on an arithmetic trick to represent fraction $\frac{S_{PAB}}{S_{QAB}}$ as the product of three other fractions, namely $\frac{S_{PAB}}{S_{PAM}}$, $\frac{S_{PAM}}{S_{QAM}}$, $\frac{S_{QAM}}{S_{QAB}}$. Then, due to axiom A10, these ratios of triangles are reduced to ratios of line segments, namely $\frac{S_{PAB}}{S_{PAM}} = \frac{\overline{AB}}{\overline{AM}}$, $\frac{S_{PAM}}{S_{QAM}} = \frac{\overline{PM}}{\overline{QM}}$, and $\frac{S_{QAM}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}$. Although short and simple, it does not provide any geometric insight.

Proof 1.

$$\frac{S_{PAB}}{S_{QAB}} = \frac{S_{PAB}}{S_{PAM}} \cdot \frac{S_{PAM}}{S_{QAM}} \cdot \frac{S_{QAM}}{S_{QAB}} = \frac{\overline{AB}}{\overline{AM}} \cdot \frac{\overline{PM}}{\overline{QM}} \cdot \frac{\overline{AM}}{\overline{AB}} = \frac{\overline{PM}}{\overline{QM}}.$$

□

In the second proof, we consider four cases represented in Fig. 3.5. They depend on whether M lies between P and Q, or A and B. We base it on Theorem VI.1.

In figures 3.6–3.10 below, we use symbols $S_1, ..., S_4$ to represent triangles. They aim to simplify formulas applied in the proof.

Proof 2.

Case 1.

Point M lies between P, Q and between A, B; see Fig. 3.6.

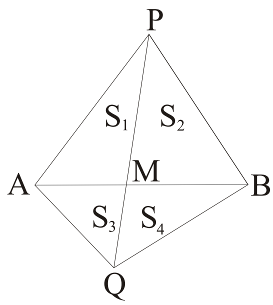


Figure 3.6: Co-side Theorem, case 1.

By Theorem VI.1, we have the following equalities (proportions)

$$\frac{S_1}{S_3} = \frac{\overline{PM}}{\overline{QM}}, \quad \frac{S_2}{S_4} = \frac{\overline{PM}}{\overline{QM}}.$$

The following equalities emulate Euclid's proportions (propositions V.12, 19), yet they can also be justified in the arithmetic of fractions:

$$\frac{S_1}{S_3} = \frac{S_2}{S_4} = \frac{S_1 + S_2}{S_3 + S_4} = \frac{\overline{PM}}{\overline{QM}}.$$

Since

$$S_1 + S_2 = S_{PAB}, \quad S_3 + S_4 = S_{QAB},$$

finally, the required equality is obtained, namely

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

Note, when $B = M$, the Co-side Theorem is the same as Euclid's VI.1 (see Fig. 3.7, namely

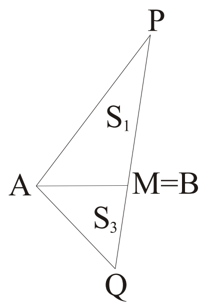


Figure 3.7: Co-side Theorem turns into VI.1.

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PB}}{\overline{BQ}} = \frac{\overline{PM}}{\overline{MQ}}.$$

Case 2.

Point M lies between A and B, but not between P and Q; see Fig. 3.8.

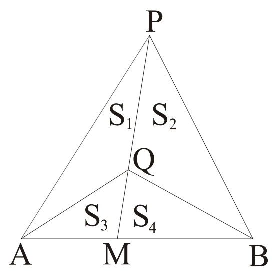


Figure 3.8: Co-side Theorem, case 2.

By Theorem VI.1

$$\frac{S_1 + S_3}{S_3} = \frac{\overline{PM}}{\overline{QM}} = \frac{S_2 + S_4}{S_4}.$$

Similarly, Euclid's theory of proportions, as well as the arithmetic of fractions justifies the following case

$$\frac{S_1 + S_2 + S_3 + S_4}{S_3 + S_4} = \frac{\overline{PM}}{\overline{QM}}.$$

Hence,

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

Case 3.

Point M lies between P and Q and not between A and B; see Fig. 3.9.

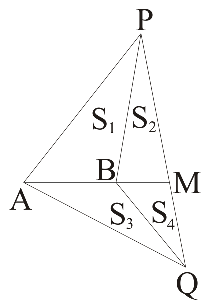


Figure 3.9: Co-side Theorem, case 3.

By Theorem VI.1

$$\frac{S_1 + S_2}{S_3 + S_4} = \frac{\overline{PM}}{\overline{QM}} = \frac{S_2}{S_4}.$$

The following equality emulates another proposition concerning Euclid's proportions (proposition V.19), yet, it also can be justified in the arithmetic of fractions.

$$\frac{S_1 + S_2 - S_2}{S_3 + S_4 - S_4} = \frac{S_1}{S_3} = \frac{\overline{PM}}{\overline{QM}}.$$

Hence,

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

Case 4.

Point M lies neither between P and Q, nor A and B; see Fig. 3.10.

By Theorem VI.1

$$\frac{S_{PAB} + S_{PBM}}{S_{QAB} + S_{QBM}} = \frac{\overline{PM}}{\overline{QM}} = \frac{S_{PBM}}{S_{QBM}}.$$

As in the previous cases,

$$\frac{S_{PAB}}{S_{QAB}} = \frac{\overline{PM}}{\overline{QM}}.$$

□

To sum up, in Euclid's VI.1, the Co-side requirement guarantees that triangles are of the same height, while their bases can differ. In the Co-side Theorem, triangles share the base,

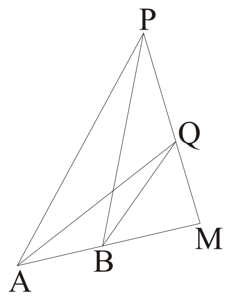


Figure 3.10: Co-side, case 4

while their heights differ. Euclid's VI.1 enables one to reduce the geometric pattern represented in Fig. 3.2 to the proportion of lines given by the formula $\frac{S_{ABC}}{S_{ADC}} = \frac{\overline{BC}}{\overline{DC}}$. The Co-side Theorem provides us with four more geometric patterns that reduce the proportions of triangles to the proportions of lines. In the next section, we show how to exploit these new patterns in school geometry.

3.4.2 Co-side Theorem in solving school geometry problems

In this section, we show how to apply the Co-side Theorem in solving school problems. For each of the four cases discussed in the previous section, we select a separate task. In these cases, we process fractions more liberally, in a way familiar to students.

Example 3.4.2 (Co-side Theorem, case 1)**.** In trapezium $ABCD$, where $AB \parallel DC$, O is the intersection point of diagonals AC and BD. Area of triangle ADB is 50. Diagonals AC and BD intersect such that $\overline{AO} : \overline{OC} = 5 : 1$. Find the area of the trapezium.

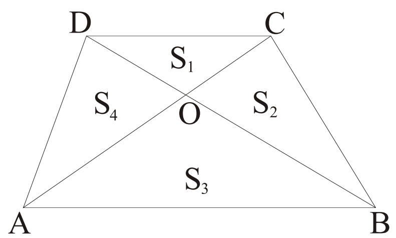


Figure 3.11: Example, Co-side Theorem case 1

To simplify our argument, we set out the following (see Fig. 3.11):

$$S_{DCO} = S_1, S_{OCB} = S_2, S_{OBA} = S_3, S_{ADO} = S_4.$$

By Co-side Theorem,

$$\frac{S_4+S_3}{S_1+S_2}=\frac{\overline{AO}}{\overline{OC}}=\frac{5}{1}.$$

Since $S_{ADB}=S_3+S_4=50$, it follows that $S_1+S_2=10$.

Hence,

$$S_{ABCD}=S_4+S_3+S_1+S_2=60.$$

□

Example 3.4.3 (Co-side Theorem, case 2)**.** *Let ABC be a triangle with a point P inside. Let lines AP and CB intersect at D, lines BP and CA – at E, and lines CP and AB – at F. Show that*

$$\frac{\overline{PD}}{\overline{AD}}+\frac{\overline{PE}}{\overline{BE}}+\frac{\overline{PF}}{\overline{CF}}=1$$

((Stankowa, Rike 2008), p. 19).

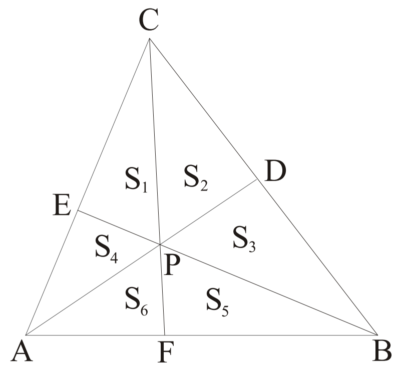


Figure 3.12: Example, Co-side Theorem case 2

We set out the following (see Fig. 3.12):

$$S_{PCE}=S_1, S_{PDC}=S_2, S_{PBD}=S_3, S_{PEA}=S_4, S_{PFB}=S_5, S_{PFA}=S_6.$$

By Co-side Theorem, we have the following equalities (proportions)

$$\frac{\overline{PD}}{\overline{AD}}=\frac{S_2+S_3}{S_{ABC}},$$

$$\frac{\overline{PE}}{\overline{BE}}=\frac{S_1+S_6}{S_{ABC}},$$

$$\frac{\overline{PF}}{\overline{CF}}=\frac{S_5+S_4}{S_{ABC}}.$$

By adding their left and right sides, we obtain

$$\frac{\overline{PD}}{\overline{AD}} + \frac{\overline{PE}}{\overline{BE}} + \frac{\overline{PF}}{\overline{CF}} = \frac{S_{ABC}}{S_{ABC}} = 1.$$

□

Example 3.4.4 (Co-side Theorem, case 3)**.** *Show that the three medians of a triangle divide it into six smaller triangles of equal area* ((Kurczab, Kurczab 2020), p. 241).

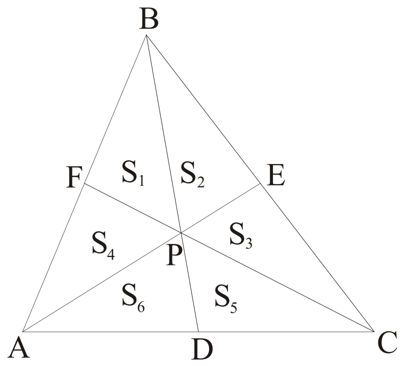


Figure 3.13: Example, Co-side Theorem case 3

Let's set out the following (see Fig. 3.13):

$$S_{PBF} = S_1, S_{PEB} = S_2, S_{PCE} = S_3, S_{PFA} = S_4, S_{PDC} = S_5, S_{PDA} = S_6.$$

The medians assumption means:

$$\overline{BF} = \overline{FA}, \quad \overline{AD} = \overline{DC}, \quad \overline{CE} = \overline{EB}$$

By Theorem VI.1

$$S_1 = S_4, \quad S_6 = S_5, \quad S_3 = S_2.$$

Applying Co-side Theorem, we get

$$\frac{S_1 + S_4}{S_2 + S_3} = \frac{\overline{AD}}{\overline{DC}} = 1,$$

which means $S_1 = S_2$.

Similarly, we show that

$$S_3 = S_5, \quad S_4 = S_6.$$

Hence,

$$S_1 = S_2 = S_3 = S_4 = S_5 = S_6.$$

$\square$

Example 3.4.5 (Co-side Theorem, case 4)**.** *Let ABC be a triangle with $\overline{BC} = a$. Let D be the midpoint of the side $\overline{AC}$, and O of line segment $\overline{BD}$. Let BC and AO intersect at P. Show that*

$$\overline{CP} = \frac{2}{3}a$$

((Babiański, Chańko, Janowicz 2019), p.14).

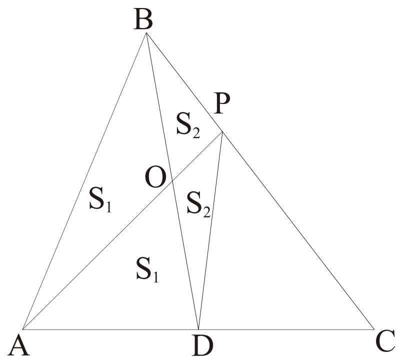


Figure 3.14: Example, Co-side Theorem case 4

Since $\overline{BO} = \overline{OD}$, by Theorem VI.1 we get

$$S_{BAO} = S_{AOD}, \quad S_{BPO} = S_{POD}.$$

Let's set out the following (see Fig. 3.14):

$$S_1 = S_{BAO} = S_{AOD} \quad S_2 = S_{BPO} = S_{POD}.$$

Since $\overline{AD} = \overline{DC}$, applying Co-side Theorem we get:

$$\frac{S_{ABP}}{S_{DBP}} = \frac{S_1 + S_2}{2 \cdot S_2} = \frac{\overline{AC}}{\overline{DC}} = \frac{2}{1}.$$

It follows that

$$S_1 = 3 \cdot S_2.$$

Again, by Co-side Theorem, we get

$$\frac{\overline{BC}}{\overline{CP}} = \frac{2 \cdot S_1}{S_1 + S_2} = \frac{3}{2}.$$

Since $\overline{BC} = a$, we finally obtain

$$\overline{CP} = \frac{2}{3}a.$$

□

3.4.3 Advanced problems

Example 3.4.6. Let D, E, F divide the sides of the triangle ABC into thirds. Draw $\overline{AE}$, $\overline{BD}$, $\overline{CF}$. Show that the area of the triangle inside, S_5, equals $\frac{1}{7}$ the area of the whole triangle. ((Hartshorne 2000), p. 212).

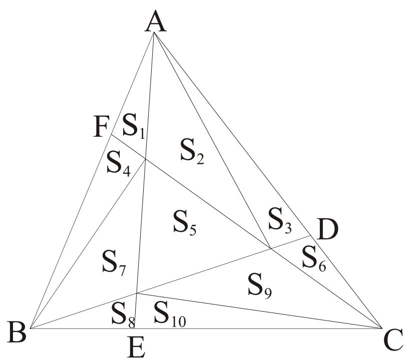


Figure 3.15: Example 3.4.6

To simplify our argument, we apply the notation as shown in Fig. 3.15. Thus

$$\overline{FB} = 2\overline{AF}, \quad \overline{DA} = 2\overline{CD} \quad \overline{EC} = 2\overline{BE}.$$

By Theorem VI.1, we have:

$$S_4 = 2S_1, \quad S_3 = 2S_6, \quad S_{10} = 2S_8.$$

Co-side Theorem, case 3, gives

$$\frac{S_5 + S_9}{S_7} = \frac{2}{1}, \quad \frac{S_5 + S_2}{S_9} = \frac{2}{1}, \quad \frac{S_5 + S_7}{S_2} = \frac{2}{1}.$$

It follows from the above that

$$S_5 + S_9 = 2S_7, \quad S_5 + S_2 = 2S_9, \quad S_5 + S_7 = 2S_2.$$

Adding up the above equations, we have

$$3S_5 = S_7 + S_9 + S_2 \tag{3.1}$$

Again, by Co-side Theorem

$$\frac{S_9 + S_8 + S_{10}}{S_3 + S_6} = \frac{2}{1}$$

Hence

$$S_9 + S_8 + S_{10} = 2(S_3 + S_6),$$

$$S_9 + S_8 + 2S_8 = 2(2S_6 + S_6),$$

$$3S_8 + S_9 = 6S_6.$$

Similarly, we have shown:

$$3S_6 + S_2 = 6S_1,$$

$$3S_1 + S_7 = 6S_8.$$

Adding up the above equations, we have

$$S_7 + S_9 + S_2 = 3(S_6 + S_1 + S_8)$$

By (3.1) we have

$$S_5 = S_6 + S_1 + S_8. \tag{3.2}$$

Note that

$$S_{ABC} = 3S_1 + 3S_8 + 3S_6 + S_5 + S_7 + S_9 + S_2.$$

By (3.1) and (3.2) we have

$$S_{ABC} = 3S_1 + 3S_8 + 3S_6 + S_6 + S_1 + S_8 + 3(S_6 + S_1 + S_8),$$

$$S_{ABC} = 7(S_6 + S_1 + S_8),$$

$$S_{ABC} = 7S_5.$$

Hence

$$S_5 = \frac{S_{ABC}}{7}.$$

□

Example 3.4.7. Let ABC be a triangle, with D lying on line AB and E on line BC, in such a way that $DE \parallel AC$. The diagonals DC and AE intersect at point F. Let lines BF and AC intersect at point G. Show that $\overline{AG} = \overline{GC}$ ((Chou, Gao, Zhang 1994), p.18).

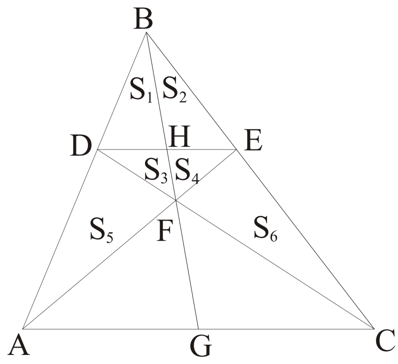


Figure 3.16: Example 3.4.7

We apply the notation as shown in Fig. 3.16.

From the assumption $DE \parallel AC$. By the definition of parallel segments, we have

$$S_4 + S_3 + S_5 = S_3 + S_4 + S_6,$$

$$S_5 = S_6. \tag{3.3}$$

It follows from *Elements*, VI.2 (Thales's Theorem)[5] that

$$\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{BE}}{\overline{EC}}. \tag{3.4}$$

By Theorem 2.5

$$\frac{S_1 + S_3}{S_5} = \frac{\overline{BD}}{\overline{DA}}, \quad \frac{S_2 + S_4}{S_6} = \frac{\overline{BE}}{\overline{EC}}. \tag{3.5}$$

By (3.4) and (3.5) we have

$$\frac{S_1 + S_3}{S_5} = \frac{S_2 + S_4}{S_6}.$$

By (3.3)

$$S_1 + S_3 = S_2 + S_4. \tag{3.6}$$

[5]"If some straight line is drawn parallel to one of the sides of a triangle then it will cut the sides of the triangle proportionally. And if the sides of a triangle are cut proportionally then the straight line joining the cutting will be parallel to the remaining side of the triangle" (Błaszczyk, Petiurenko 2019, 52).

Co-side Theorem (case 1) gives

$$\frac{S_1 + S_3}{S_2 + S_4} = \frac{\overline{DH}}{\overline{HE}},$$

Hence, $\overline{DH} = \overline{HE}$.

By Co-side Theorem (case 3)

$$\frac{S_1 + S_3 + S_5}{S_2 + S_4 + S_6} = \frac{\overline{AG}}{\overline{GC}}$$

By (3.6) and (3.3)

$$S_1 + S_3 + S_5 = S_2 + S_4 + S_6.$$

Hence,

$$\overline{AG} = \overline{GC}.$$

□

Chapter 4

Automatic Theorem Proving Based on the Area Method

4.1 Elimination Lemmas

Lemma 4.1.1. ***(EL1)***

(The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$), then:

$$\frac{\overline{PM}}{\overline{QM}} = \frac{S_{PAB}}{S_{QAB}}$$

(Quaresma, Janičić 2006)

Lemma 4.1.2. ***(EL1*)***

(The Co-side Theorem) Let M be the intersection of two non-parallel lines AB and PQ and $Q \neq M$), then:

$$\frac{\overline{PQ}}{\overline{PM}} = \frac{S_{AQBP}}{S_{ABP}}$$

(Quaresma, Janičić 2006)

Proof.

$$\frac{\overline{PM}}{\overline{PQ}} = \frac{S_{PMB}}{S_{PQB}}, \; \frac{\overline{PM}}{\overline{PQ}} = \frac{S_{PAM}}{S_{APQ}}$$

$$S_{PQB} = \frac{\overline{PQ} \cdot S_{PMB}}{\overline{PM}}, \; S_{APQ} = \frac{\overline{PQ} \cdot S_{PAM}}{\overline{PM}}$$

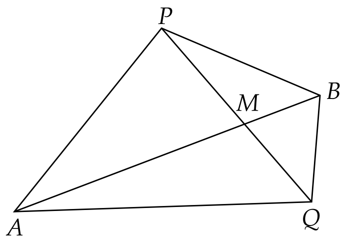


Figure 4.1: EL1*

$$S_{AQBP} = S_{PQB} + S_{APQ} = \frac{\overline{PQ}}{\overline{PM}}(S_{PMB} + S_{PAM}) = \frac{\overline{PQ}}{\overline{PM}} \cdot S_{PAB} \Rightarrow$$
$$\Rightarrow \frac{\overline{PQ}}{\overline{PM}} = \frac{S_{AQBP}}{S_{ABP}}$$

□

Lemma 4.1.3. ***(EL2)*** *Let Y be introduced by intersection line PQ and AY, and let $AY \parallel CD$, then:*

$$\frac{\overline{AY}}{\overline{CD}} = \frac{S_{APQ}}{S_{CPDQ}}.$$

(Quaresma, Janičić 2006)

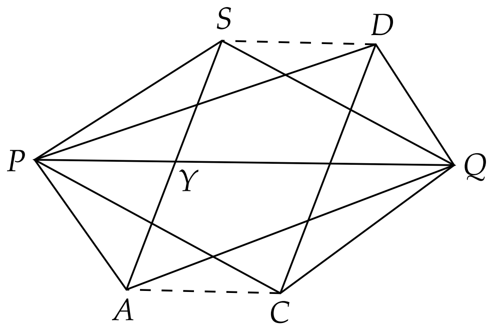


Figure 4.2: EL2

Proof. We construct point S on the line AY and $\overline{AS} = \overline{CD}$. We have:

$$\frac{\overline{AY}}{\overline{CD}} = \frac{\overline{AY}}{\overline{AS}}.$$

By lemma 4.1.2

$$\frac{\overline{AY}}{\overline{AS}} = \frac{S_{APQ}}{S_{APSQ}}.$$

By the axiom (Euclid's Theorem VI.1)

$$\frac{S_{APQ}}{S_{APSQ}} = \frac{S_{APQ}}{S_{CPDQ}}.$$

$\square$

Lemma 4.1.4. ***(EL3)***
Let M be a point such that $\frac{\overline{PM}}{\overline{UV}} = r$, then for any points X and Y on lines PM and UV we have:

$$S_{XYM} = S_{XYP} + r(S_{XYV} - S_{XYU})$$

(Quaresma, Janičić 2006)

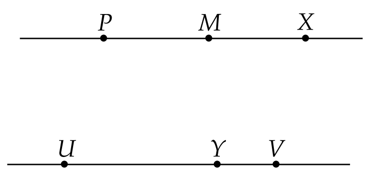


Figure 4.3: EL3

Proof. The case when X and Y are on the same line PM or UV is trivial. We consider the case when $X \in CM$, and $Y \in UV$. By Axiom 10 we have:

$$\frac{\overline{MX}}{\overline{PX}} = \frac{S_{XYM}}{S_{XYP}}$$

$$S_{XYM} = \frac{\overline{MX} \cdot S_{XYP}}{\overline{PX}} = \frac{(\overline{PX} - \overline{PM}) \cdot S_{XYP}}{\overline{PX}} = S_{XYP} - \frac{\overline{PM}}{\overline{PX}} \cdot S_{XYP}$$

The triangles XYC and UXV have the same height. By Theorem VI.1 we have:

$$\frac{\overline{PX}}{\overline{UV}} = \frac{S_{XYP}}{S_{UXV}} \Rightarrow \overline{PX} = \frac{\overline{UV} \cdot S_{XYP}}{S_{UXV}}$$

$$S_{XYM} = S_{XYP} - \frac{\overline{PM} \cdot S_{XYP} \cdot S_{UXV}}{\overline{UV} \cdot S_{XYP}} = S_{XYP} - \frac{\overline{PM}}{\overline{UV}} \cdot S_{UXV} =$$

$$S_{XYP} - r(S_{XUY} + S_{XYV}) = S_{XYP} - r(S_{XYV} + S_{XYU})$$

$\square$

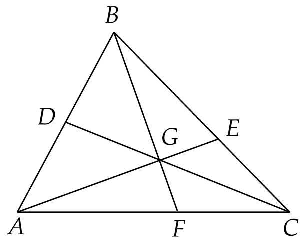


Figure 4.4

Example 4.1.5 (Ceva's Theorem)**.** Let ABC be a triangle and D, E, F are points on straight lines AB, BC and CA respectively, so that segments $\overline{AE}$, $\overline{BF}$ and $\overline{CD}$ are intersected at the point G. Show that:

$$\frac{\overline{AF}}{\overline{FC}} \cdot \frac{\overline{CE}}{\overline{EB}} \cdot \frac{\overline{BD}}{\overline{DA}} = 1$$

Proof. Points construction:

	CG, AB	AG, BC	BG, AC
A, B, C, G	D	E	F

By the statement:

$$\frac{\overline{AF}}{\overline{FC}} \cdot \frac{\overline{CE}}{\overline{EB}} \cdot \frac{\overline{BD}}{\overline{DA}} = 1$$

Eliminate point F by the Lemma 4.1.1:

$$\frac{S_{AGB}}{S_{CBG}} \cdot \frac{\overline{CE}}{\overline{EB}} \cdot \frac{\overline{BD}}{\overline{DA}} = 1$$

Eliminate point E by the 4.1.1:

$$\frac{S_{AGB}}{S_{CBG}} \cdot \frac{S_{AGC}}{S_{BAG}} \cdot \frac{\overline{BD}}{\overline{DA}} = 1$$

Eliminate point D by the 4.1.1:

$$\frac{S_{AGB}}{S_{CBG}} \cdot \frac{S_{AGC}}{S_{BAG}} \cdot \frac{S_{BGC}}{S_{CAG}} = 1$$

By the geometric and algebraic simplification, we have:

$$1 = 1$$

Trivial equality. $\square$

Example 4.1.6. If two parallel lines cut from the sides of an arbitrary angle the segments AY, YS and AH, HD respectively, then we always have the proportion

$$\frac{\overline{BD}}{\overline{AD}} = \frac{\overline{CE}}{\overline{AE}}$$

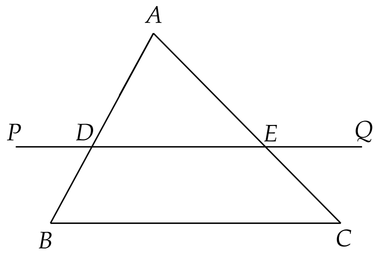


Figure 4.5: Example of EL2 and EL3

Proof. Points construction:

	$BC \parallel P$	AB, PQ	AC, PQ
A, B, C, P	Q	D	E

By the statement:

$$\frac{\overline{BD}}{\overline{AD}} = \frac{\overline{CE}}{\overline{AE}}$$

Eliminate point E by the Lemma 4.1.4:

$$\frac{\overline{BD}}{\overline{AD}} = \frac{S_{APQ}}{S_{APCQ}}$$

Eliminate point D by the Lemma 4.1.4:

$$\frac{S_{APQ}}{S_{APBQ}} = \frac{S_{APQ}}{S_{APCQ}}$$

By the algebraic simplification:

$$S_{APCQ} = S_{APBQ}$$

Eliminate point Q from the left part by the Lemma 4.1.4:

$$S_{APCP} + 1 \cdot (S_{APCC} + (-1 \cdot S_{APCB})) = S_{APBQ}$$

By the algebraic simplification:

$$S_{APCP} + S_{APCC} + (-1 \cdot S_{APCB}) = S_{APBQ}$$

Eliminate point Q from the right part by the Lemma 4.1.4:

$$S_{APCP} + S_{APCC} + (-1 \cdot S_{APCB}) = S_{APBP} + 1 \cdot (S_{APBC} + (-1 \cdot S_{APBB}))$$

By the algebraic simplification:

$$S_{APCP} + S_{APCC} + (-1 \cdot S_{APCB}) = S_{APBP} + S_{APBC} + (-1 \cdot S_{APBB})$$

By the geometric simplification:

$$(S_{APC} + (-1 \cdot S_{APC})) + (S_{APC} + 0) + (-1 \cdot (S_{APC} + S_{ACB})) =$$

$$= (S_{APB} + (-1 \cdot S_{APB})) + (S_{APB} + (-1 \cdot S_{ACB})) + (-1 \cdot (S_{APB} + 0))$$

By the algebraic simplification:

$$0 = 0$$

Trivial equality.

□

Example 4.1.7. Let $ABCD$ and $EBCF$ be parallelograms on the same base BC, and between the same parallels AF and BC. I say that $ABCD$ is equal to parallelogram $EBCF$.

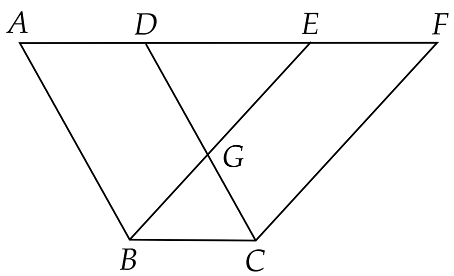


Figure 4.6: EL3 example

Proof. Points construction:

	$\overline{BC} \parallel A$	AD	$\overline{BC} \parallel E$
A, B, C	D, $(\overline{AD} = \overline{BC})$	E	F, $(\overline{EF} = \overline{BC})$

It is enough to show that $S_{CAD} = S_{BEF}$.

By the statement:

$$S_{CAD}{=}S_{BEF}$$

Eliminate point F by the Lemma 4.1.4:

$$S_{CAD}{=}S_{BEE} + 1 \cdot (S_{BEC} + (-1 \cdot S_{BEB}))$$

By the geometric simplifications:

$$S_{CAD}{=}0 + 1 \cdot (S_{CBE} + (-1 \cdot 0))$$

By the algebraic simplifications:

$$S_{CAD}{=}S_{CBE}$$

Eliminate point E by the Lemma 4.1.4:

$$S_{CAD}{=}S_{CBA} + r_0 \cdot (S_{CBD} + (-1 \cdot S_{CBA}))$$

By the algebraic simplifications:

$$S_{CAD}{=}S_{CBA} + r_0 \cdot S_{CBD} + (-1 \cdot r_0 \cdot S_{CBA})$$

Eliminate point D from the left part by the Lemma 4.1.4:

$$S_{CAA} + 1 \cdot (S_{CAC} + (-1 \cdot S_{CAB})){=}S_{CBA} + r_0 \cdot S_{CBD} + (-1 \cdot (r_0 \cdot S_{CBA}))$$

By the geometric simplifications:

$$0 + 1 \cdot (0 + (-1 \cdot S_{CAB})){=}{-1} \cdot S_{CAB} + r_0 \cdot S_{CBD} + (-1 \cdot (r_0 \cdot (-1 \cdot S_{CAB})))$$

By the algebraic simplifications:

$$0{=}S_{CBD} + S_{CAB}$$

Eliminate point D from the right part by the Lemma 4.1.4:

$$0{=}S_{CBA} + 1 \cdot (S_{CBC} + (-1 \cdot S_{CBB})) + S_{CAB}$$

By the geometric simplifications:

$$0{=}S_{CBA} + 1 \cdot (0 + (-1 \cdot 0)) + (-1 \cdot S_{CBA})$$

By the algebraic simplifications:

$$0=0$$

Trivial equality.

□

4.2 Theorem Prover GCLC

The GCLC is a tool for teaching geometry, visualization and creating mathematical illustrations, as well as automatically proving geometric theorems (GCLC 2022). Theorem Prover GCLC generates traditional proofs based on geometric properties of objects, not their coordinates. In this sense, it is automatically proved by theorems in synthetic geometry, which justifies its application to Euclid.

Property	In terms of geometric quantities
Points A and B are identical	$P_{ABA} = 0$
Points A, B, C are collinear	$S_{ABC} = 0$
AB is perpendicular to CD	$P_{ABA} \neq 0 \wedge P_{CDC} \neq 0 \wedge P_{ACD} = P_{BCD}$
AB is parallel to CD	$P_{ABA} \neq 0 \wedge P_{CDC} \neq 0 \wedge S_{ACD} = S_{BCD}$
O is the midpoint of AB	$S_{ABO} = 0 \wedge P_{ABA} \neq 0 \wedge \frac{AO}{AB} = \frac{1}{2}$
AB has the same length as CD	$P_{ABA} = P_{CDC}$
Points A, B, C, D are harmonic	$S_{ABC} = 0 \wedge S_{ABD} = 0 \wedge P_{BCB} \neq 0 \wedge P_{BDB} \neq 0 \wedge \frac{AC}{CB} = \frac{DA}{DB}$
Angle ABC has the same measure as DEF	$P_{ABA} \neq 0 \wedge P_{ACA} \neq 0 \wedge P_{BCB} \neq 0 \wedge P_{DED} \neq 0 \wedge P_{DFD} \neq 0 \wedge P_{EFE} \neq 0 \wedge S_{ABC} \cdot P_{DEF} = S_{DEF} \cdot P_{ABC}$
A and B belong to the same circle arc CD	$S_{ACD} \neq 0 \wedge S_{BCD} \neq 0 \wedge S_{CAD} \cdot P_{CBD} = S_{CBD} \cdot P_{CAD}$

Table 4.1: Expressing geometry predicates in terms of the three geometric quantities (Janičić, Narboux, Quaresma 2012)

Constructs that can be expressed in the form of `INTER`, `PRATIO`, `TRATIO` and `FOOT` commands are allowed in prover:

`INTER Y ln1 ln2`: Point Y is the intersection of line $ln1$ and line $ln2$.

`PRATIO Y W U V r`: Point Y is a point such that $\overline{WY} = r\overline{UV}$, where r is a real number.

`TRATIO Y U V r`: Point Y is a point on line l, such that $r = \frac{UY}{UV}$, where r is a real number and l is a line such that U lies on l and l is perpendicular to UV.

`FOOT Y P U V`: Point Y is a foot from P to line UV (i.e., YP is perpendicular to UV and Y lies on UV. For each point X constructed by the above constructions and for each geometry quantity g involving X, there is a suitable lemma that enables replacing g by an expression with no occurrences of X. Thanks to these lemmas, all constructed points can be eliminated from the conjecture (GCLC manual 2022).

ratio of directed segments	$\frac{\overline{PQ}}{\overline{AB}}$	`sratio P Q A B`
signed area (arity 3)	S_{ABC}	`signed_area3 A B C`
signed area (arity 4)	S_{ABCD}	`signed_area4 A B C D`
Pythagoras difference (arity 3)	P_{ABC}	`pythagoras_difference3 A B C`
Pythagoras difference (arity 4)	P_{ABCD}	`pythagoras_difference4 A B C D`

Table 4.2: Geometry quantities in GCLC
(GCLC manual 2022)

points A and B are identical	`identical A B`
points A, B, C are collinear	`collinear A B C`
AB is perpendicular to CD	`perpendicular A B C D`
AB is parallel to CD	`parallel A B C D`
O is the midpoint of AB	`midpoint O A B`
AB has the same length as CD	`same_length A B C D`
points A, B, C, D are harmonic	`harmonic A B C D`

Table 4.3: Statements for the basic sorts of conjectures in GCLC
(GCLC manual 2022)

$=$	`equality`
$+$	`sum`
$\cdot$	`mult`
$/$	`ratio`

Table 4.4: Operators in textual form in GCLC
(GCLC manual 2022)

4.3 Automatic Proving of Euclid's Book VI

Theorem 4.3.1 (VI.1)**.** *Let ABC and ACD be triangles, and EC and CF parallelograms, of the same height AC. I say that as base BC is to base CD, so triangle ABC (is) to triangle ACD, and parallelogram EC to parallelogram CF.*

Proof. Case 1. Proof that as base BC is to base CD, so triangle ABC is to triangle ACD.

Construction steps:

```
point B 20 20
point C 60 20
point A 45 50
online D C B
```

Construction command `online` is expressed in terms of `PRATIO`. It introduces new points, and introduces an indeterminate constant r: for instance `online X A B` is interpreted as `PRATIO X A A B r`.

The coordinates of the entered points are not taken into account in the automatic proof. Point, area, segment are primitive concepts; they do not have any numerical expressions in the automatic proof; they are symbols.

Theorem thesis in terms of automatic proof:

```
prove { equal { sratio B C C D } { ratio { signed_area3 B C A } {
    signed_area3 C D A } } }
```

In ordinary mathematical language it can be written like this:

$$\frac{\overline{BC}}{\overline{CD}} = \frac{S_{BCA}}{S_{CDA}}.$$

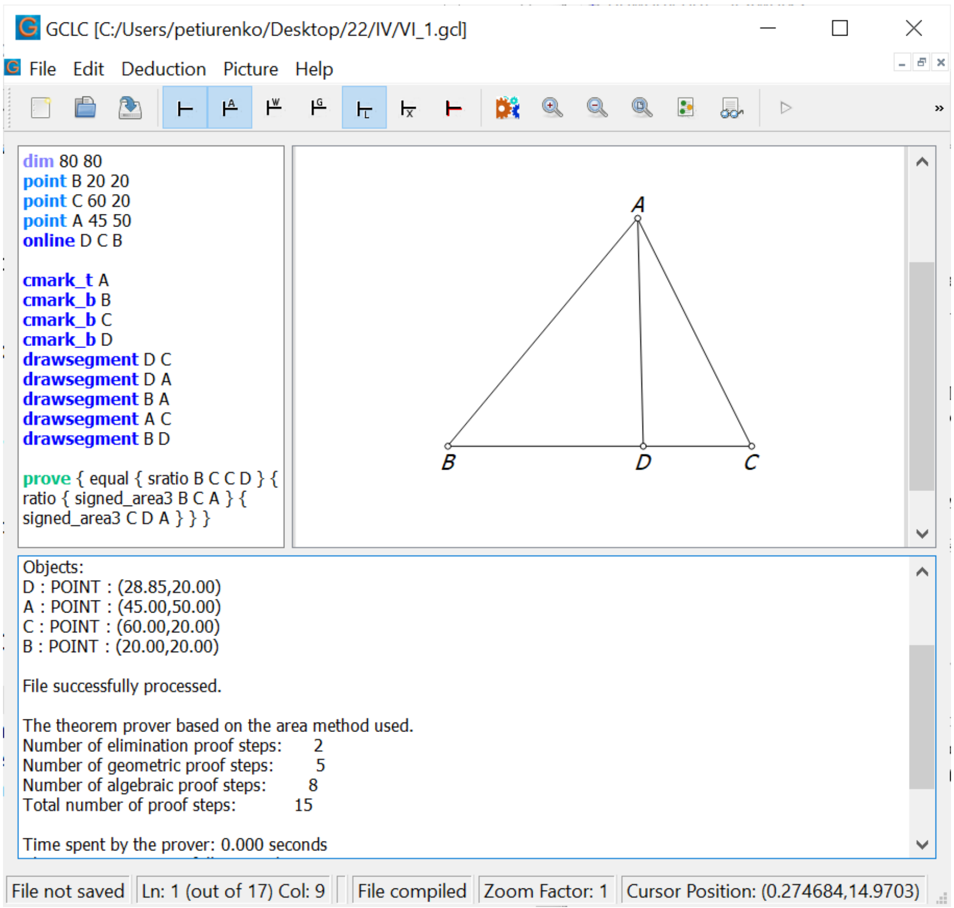


Figure 4.7: Work window of GCLC for case 1 of theorem VI.1

After the automatic proof, we receive the message that the theorem has been successfully proven and NDG conditions are: $P_{CDC} \neq 0$ i.e., points C and D are not identical.

If we want to visualize our theorem, we need to add the necessary drawing elements. It will be shown only in the example of one theorem. Visualization helps us to illustrate the proposition, but the constructed points, lines, etc. have no relation to the proof.

Construction steps:

```
cmark_t A
cmark_b B
cmark_b C
cmark_b D
drawsegment D C
drawsegment D A
```

```
drawsegment B A
drawsegment A C
drawsegment B D
```

Case 2. Proof that as base BC is to base CD, so parallelogram EC is to parallelogram CF.

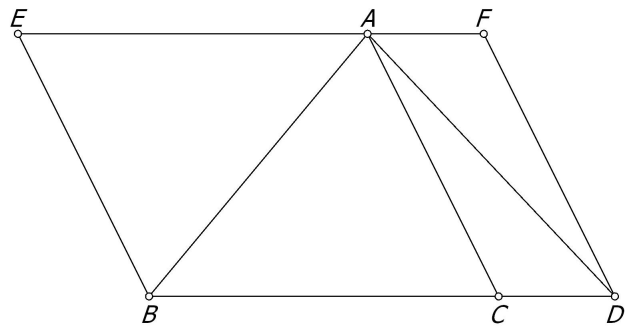


Figure 4.8: Theorem VI.1 case 2, construction at GCLC

□

We add new construction steps to the already introduced ones to obtain parallelograms.

Construction steps:

```
line bc B C
parallel a~A bc
line ac A C
parallel be B ac
parallel df D ac
intersec E be a
intersec F df a
```

We have new construction commands:

`parallel` introduce one auxiliary point: for instance, `parallel p A q` introduces a point P_p on a line parallel to q; the line p is then determined by the points A and P_p;

`line` define a line that passes through two points;

`intersec` introduces the point of intersection of two lines.

Theorem thesis for case 2 in terms of automatic proof:

```
prove { equal { sratio B C C D } {  ratio { signed_area4 B E A C } {
    signed_area4 C A F D } } }
```

In ordinary mathematical language it can be written like this:

$$\frac{\overline{BC}}{\overline{CD}} = \frac{S_{BEAC}}{S_{CAFD}}.$$

Automatic proof is quite complicated, but we have a final score of $0 = 0$, which is evidence of a successful proof.

NDG conditions are:

$S_{BAP_a^1} \neq S_{P_{be}^2 AP_a^1}$ i.e., lines BP_{be}^2 and AP_a^1 are not parallel;

$S_{DAP_a^1} \neq S_{P_{df}^3 AP_a^1}$ i.e., lines DP_{df}^3 and AP_a^1 are not parallel;

$P_{CDC} \neq 0$ i.e., points C and D are not identical;

$S_{BAC} \neq 0$ i.e., points B, A and C are not collinear;

$r_0 \neq 0$ i.e., constant r_0 is non-zero.

The proof will remain the same.Points P_a^1, P_{be}^2 and P_{df}^3 from NDG are additional points introduced by the GCLC prover during the construction of parallel lines.

Theorem 4.3.2 (VI.2)**.** *Let DE be drawn parallel to one of the sides BC of triangle ABC. I say that as BD is to DA, so CE is to EA. Let the sides AB and AC of triangle ABC be cut proportionally such that as BD is to DA, so CE is to EA. Let DE be joined. I say that DE is parallel to BC.*

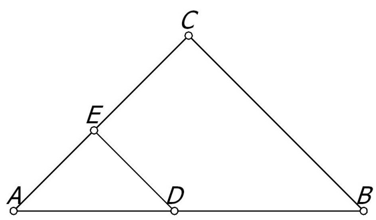


Figure 4.9: Theorem VI.2, construction at GCLC

Proof. Case 1. Let DE be drawn parallel to one of the sides BC of triangle ABC. We need to show that as BD is to DA, so CE is to EA. Construction steps:

```
point A 20 30
point B 60 30
point C 40 50
online D A B
line bc B C
line ca C A
parallel de D bc
intersec E ca de
```

Theorem thesis for case 2 in terms of automatic proof:

```
prove { equal { sratio B D D A } { sratio C E E A } } }
```

In ordinary mathematical language it can be written like this:

$$\frac{\overline{BD}}{\overline{DA}} = \frac{\overline{CE}}{\overline{EA}}.$$

NDG conditions are:

$S_{CDP_{de}^1} \neq S_{ADP_{de}^1}$ i.e., lines CA and DP_{de}^1 are not parallel;

$P_{DAD} \neq 0$ i.e., points D and A are not identical;

$P_{EAE} \neq 0$ i.e., points E and A are not identical;

$S_{CAB} \neq 0$ i.e., points C, A and B are not collinear;

$r_0 \neq 0$ i.e., constant r_0 is non-zero.

Case 2. Let the sides AB and AC of triangle ABC be cut proportionally such that as BD is to DA, so CE is to EA. Let DE have been joined. Prove that DE is parallel to BC.

This case is a bit more complicated. We have the following construction steps:

```
point A 20 30
point B 60 30
point C 40 50
towards D A B 0.3
towards E A C 0.3
```

We use new construction command:

`towards` is expressed in terms of `PRATIO`, it does not introduce new points.

Theorem thesis for case 2 in terms of automatic proof:

```
prove { parallel D E B C }
```

In the towards command we introduce a concrete 0.3 parameter which means that $\frac{AD}{DB} = \frac{AE}{EC} = \frac{1}{3}$. This may mean that the proof we get is not general. Let's analyze the automatic proof GCLC:

$$S_{DBC}{=}S_{EBC}$$

$$S_{DBC}{=}S_{BCE}$$

$$S_{DBC}{=}(S_{BCA} + (0.3 \cdot (S_{BCC} + (-1 \cdot S_{BCA}))))$$

$$S_{BCD}{=}(S_{BCA} + (0.3 \cdot (0 + (-1 \cdot S_{BCA}))))$$

$$S_{BCD}{=}(0.7 \cdot S_{BCA})$$

$$(S_{BCA} + (0.3 \cdot (S_{BCB} + (-1 \cdot S_{BCA})))){=}(0.7 \cdot S_{BCA})$$

$$(S_{BCA} + (0.3 \cdot (0 + (-1 \cdot S_{BCA})))){=}(0.7 \cdot S_{BCA})$$

$$0 = 0$$

If we change 0.3 to r and 0.7 to $1 - r$, where $r \in \mathbb{R}$, the proof will not change, and hence proof is general.

□

Note: If we look at the first line of the proof, we can see that the prover reformulated the parallelity to the equality of two areas of triangles (because this is exactly how the parallelity of two lines is defined in the area method). Hence, the thesis of theorem VI. 1 case 2 can be formulated as

```
proof {equal {signed_area3 DBC} {signed_area3 EBC}}.
```

The proof will remain the same.

Theorem 4.3.3 (VI.3)**.** *Let ABC be a triangle. Let the angle BAC be cut in half by the straight line AD. I say that as BD is to CD, so BA is to AC.*

Proof. In the case of theorem VI.3 we have a problem because the GCLC prover does not allow the bisector tool to be used. We cannot construct a bisector in a defined angle, but we can construct "some" angle with the bisector.

We have next construction steps:

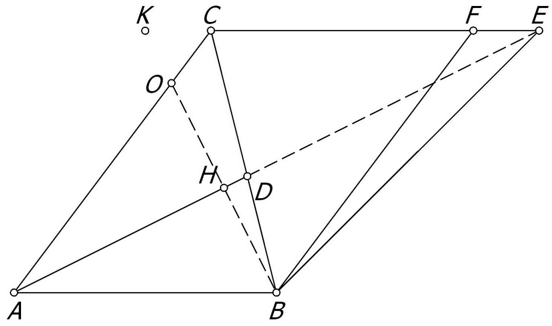


Figure 4.10: Theorem VI.3, construction at GCLC

```
point A 40 30
point B 70 30
point K 55 60
line kb K B
foot H A kb
translate O B H H
online C O A
line ah A H
line bc B C
intersec D ah bc
line ab A B
parallel cd C ab
line ac A C
parallel bf B ac
intersec E ah cd
intersec F bf cd
```

We will describe the above construction in more detail. Let us have any of the points A, B and K. Construct a perpendicular AH from point A to the segment BK. Then we construct a point O symmetrical to B with respect to H. For this we will use the new construction command `translate` (expressed in terms of `PRATIO` , it does not introduce new points). Triangle BAO is isosceles and AH is a bisector of $\angle A$. Choose any point C on line AO. The point D is the intersection of AH and BC, and AD is the bisector of angle CAB. The ratio command can only be used on parallel segments. In the case of AB and AC, there are no parallel segments. We make additional constructs (See Fig. 4.10):

$ABFC$ is a parallelogram $\Rightarrow AB = CF$,

triangle ACE is isosceles $\Rightarrow CA = CE$.

Finally, we can formulate the thesis in terms of the GCLC:

```
prove { equal { sratio B D C D } { sratio  F C C E} }
```

□

Note: Above we have discussed the case where AD is the bisector of an inside angle of the triangle, but in GCLC we immediately get the theorem for the case of the bisector of the triangle's outside angle too (see Fig. 4.11).

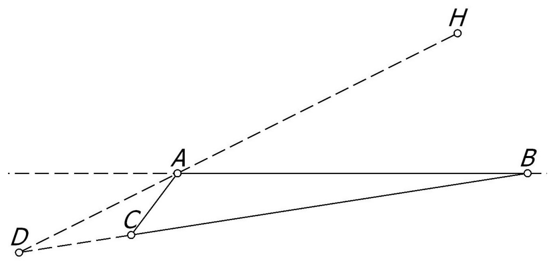


Figure 4.11: Theorem VI.3, construction at GCLC (case for bisector of outside angle)

Theorem 4.3.4 (VI.4). *Let ABC and DCE be equiangular triangles, having angle ABC equal to DCE, and angle BAC to CDE, and further, angle ACB to CED. I say that in triangles ABC and DCE the sides about the equal angles are proportional, and those sides subtending equal angles correspond.*

Proof. Construction steps:

```
point A 20 20
point C 50 20
point B 25 55
online E A C
line bc B C
line ab A B
```

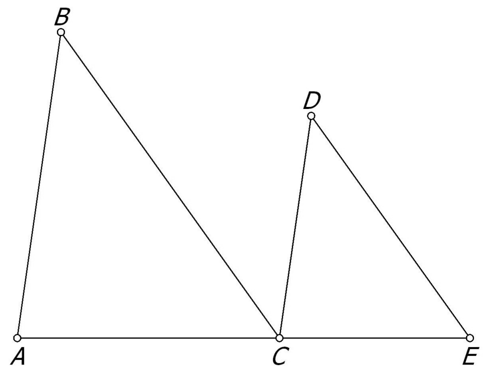


Figure 4.12: Theorem VI.4, construction at GCLC

```
parallel ed E bc
parallel cd C ab
intersec D ed cd
```

Theorem thesis in terms of automatic proof:

```
prove { equal { sratio A B C D} { sratio B C D E} }
```

□

Theorem 4.3.5 (VI.5)**.** *Let ABC and DEF be two triangles having proportional sides, so that as AB is to BC, so DE is to EF, and as BC is to CA, so EF is to FD, and further, as BA is to AC, so ED is to DF. I say that triangle ABC is equiangular to triangle DEF, and that the triangles will have the angles which corresponding sides subtend equally. That is, angle ABC is equal to DEF, BCA to EFD, and further, BAC to EDF.*

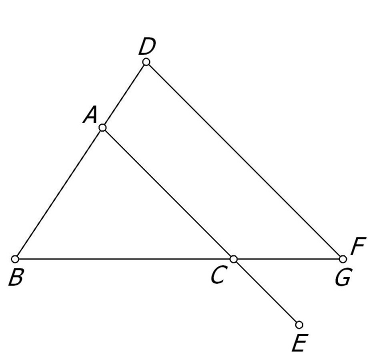


Figure 4.13: Theorem VI.5, construction at GCLC

Proof. Let us have triangle ABC. We need to construct a triangle with all sides in proportion to the sides of ABC. Construction steps:

```
point B 60 40
point C 85 40
point A 70 55
line bc B C
line ac A C
line ab A B
towards D B A 1.5
towards F B C 1.5
towards E A C 1.5
translate G  A E D
```

Comment to the construction (see Fig. 4.13). We construct three segments BD, BF and AE so that $\frac{BA}{BD} = \frac{BC}{BF} = \frac{AC}{AE}$. Next, we translate point D to a vector AE and get point G. Now, if we prove that point F and point G are identical, then the obtained triangle DBF will be a triangle with sides which are proportional to the sides of ABC. We enter the appropriate command in the prover:

```
prove { identical F G }
```

We get a positive answer from the prover. Next, we can prove the main thesis of Theorem VI.5. Firstly we show $\angle BAC = \angle BDF$:

```
prove { equal { ratio { pythagoras_difference3 B A C }    {
    signed_area3 B A C } } { ratio { pythagoras_difference3 B D F } {
      signed_area3 B D F } } }
```

We have in ordinary mathematical language:

$$\frac{P_{BAC}}{S_{BAC}} = \frac{P_{BDF}}{S_{BDF}}$$

And analogously we prove the equality of angles ACB and DFB:

```
prove { equal { ratio { pythagoras_difference3 A C B }    {
    signed_area3 A C B } } { ratio { pythagoras_difference3 D F B } {
      signed_area3 D F B } } }
```

□

Theorem 4.3.6 (VI.6)**.** *Let ABC and DEF be two triangles having one angle, equal to one angle, EDF (respectively), and then as BA is to AC, so ED is to DF. I say that triangle ABC is equiangular to triangle DEF, and will have angle ABC equal to DEF, and angle ACB to DFE.*

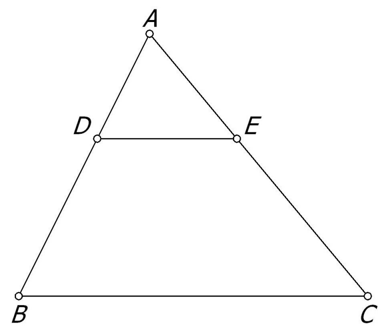


Figure 4.14: Theorem VI.6, construction at GCLC

Proof. Construction steps:

```
point A 35 50
point B 20 20
point C 60 20
towards D A B 0.4
towards E A C 0.4
```

Firstly we show $\angle ABC = \angle ADE$:

```
prove { equal { ratio { pythagoras_difference3 A B C }  {
  signed_area3 A B C } } { ratio { pythagoras_difference3 A D E } {
   signed_area3 A D E} } }
```

Next, we show $\angle ACB = \angle AED$:

```
prove { equal { ratio { pythagoras_difference3 A C B }  {
  signed_area3 A C B } } { ratio { pythagoras_difference3 A E D} {
  signed_area3 A E D } } }
```

In mathematical language we have:

$$\frac{P_{ABC}}{S_{ABC}}=\frac{P_{ADE}}{S_{ADE}} \text{ and } \frac{P_{ACB}}{S_{ACB}}=\frac{P_{AED}}{S_{AED}}.$$

In both cases, the prover successfully proves these.

□

Theorem 4.3.7 (VI.7). *Let ABC and DEF be two triangles having one angle BAC, equal to one angle EDF respectively, and the sides about some other angles, ABC and DEF respectively, proportional, so that as AB is to BC, so DE is to EF, and the remaining angles at C and F, first of all, both less than right-angles. I say that triangle ABC is equiangular to triangle DEF, and that angle ABC will be equal to DEF, and that the remaining angle at C will be manifestly equal to the remaining angle at F.*

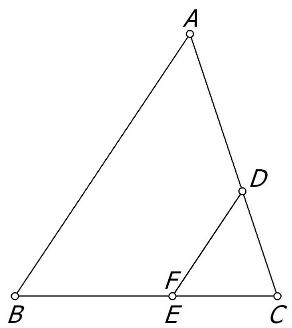


Figure 4.15: Theorem VI.7, construction at GCLC

Proof. Construction steps:

```
point A 30 40
point B 10 10
point C 40 10
towards D C A 0.4
towards E C B 0.4
line ab A B
line bc B C
parallel df D ab
intersec F df bc
```

Comment to the construction: firstly we construct points D and E so that $\frac{AC}{DC} = \frac{BC}{EC}$. Next, we construct $DF \parallel AB$ and check if point E is identical to point F `prove { identical E F }` . We get a positive answer, so $\angle EDC = \angle BAC$.

We need to show $\angle ABC = \angle DEC$:

```
prove { equal { ratio { pythagoras_difference3 A B C }   {
    signed_area3 A B C } } { ratio { pythagoras_difference3 D E C } {
     signed_area3 D E C } } }
```

The prover successfully proves these.

□

Note: When we use automatic proof of theorem VI.7, we do not need to discuss two different cases: 1) two acute angles; 2) acute and obtuse angles. Since the points A, B, C are arbitrary points in the plane, the prover takes all cases into account.

Theorem 4.3.8 (VI.8)**.** *Let ABC be a right-angled triangle having the angle BAC a right-angle, and let AD be drawn from A, perpendicular to BC. I say that triangles ABD and ADC are each similar to the whole triangle ABC and further, to one another.*

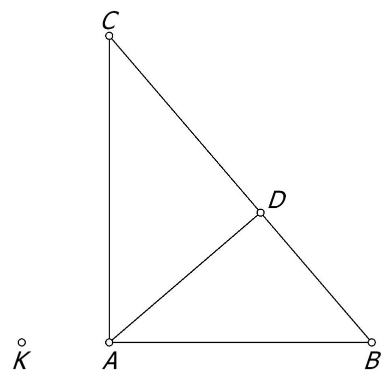


Figure 4.16: Theorem VI.8, construction at GCLC

Proof. Construction steps:

```
point K 20 20
point B 60 20
point C 30 60
```

```
line ab K B
foot A C ab
line cb C B
foot D A cb
```

Comment on the construction. Firstly, we need to construct a rectangular triangle ABC. We construct any points C, B and K. The K point is the auxiliary point, thanks to which we define the line ab. Next, we use the new construction command `foot` :

`foot` is expressed in terms of `FOOT`, it does not introduce new points.

With the foot tool we get point A (ABC - right triangle) and point D ($AD \perp BC$).

To prove the theorem, it is best to show that the theorem's triangles are equiangular:

1. $\angle ACB = \angle BAD$

```
prove { equal { ratio { pythagoras_difference3 A C B } {
    signed_area3 A C B } } { ratio { pythagoras_difference3 B A D } {
      signed_area3 B A D } } }
```

2. $\angle CBA = \angle DAC$

```
prove { equal { ratio { pythagoras_difference3 C B A } {
    signed_area3 C B A } } { ratio { pythagoras_difference3 D A C } {
      signed_area3 D A C } } }
```

The prover successfully proves these.

□

Theorem 4.3.9 (VI.9). *Let AB be the given straight line. It is required to cut off a prescribed part from AB.*

Proof. Construction steps:

```
point A 20 20
point B 60 20
point D 30 30
towards C A D 3
line cb C B
```

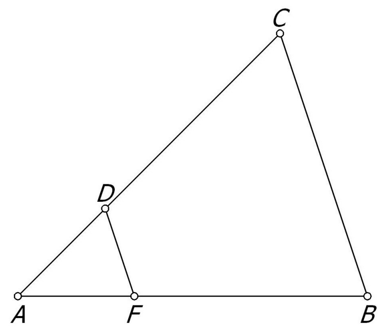


Figure 4.17: Theorem VI.9, construction at GCLC

```
parallel df D cb
line ab A B
intersec F df ab
```

Comment on the construction. The command 'towards' allows us to postpone the segment AD on the line as many times as we want (three times in theexample) and then repeat Euclid's construction. Next, we need to prove that the cut-off part is correct:

```
prove { equal { sratio A B A F } { 3 } }
```

The prover successfully proves these.

□

Theorem 4.3.10 (VI.10)**.** *To cut a given uncut straight line similarly to a given cut straight-line.*

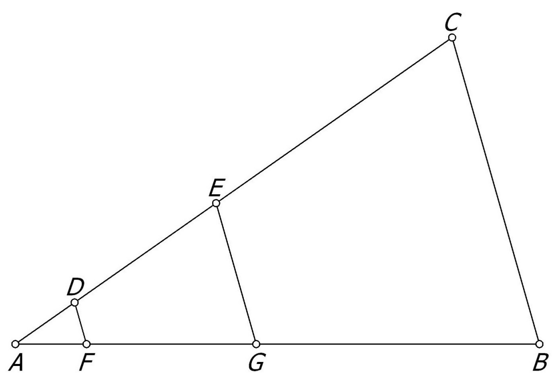


Figure 4.18: Theorem VI.10, construction at GCLC

Proof. Construction steps:

```
point A 20 20
point B 80 20
point C 70 55
online D A C
online E A C
line cb C B
parallel df D cb
parallel eg E cb
line ab A B
intersec F ab df
intersec G ab eg
```

We completely repeat Euclid's construction and introduce the thesis for proof into the prover:

```
prove { equal { sratio C E E D} { sratio B G G F } }
```

and

```
prove { equal { sratio E D D A} { sratio G F F A} }
```

The conjecture is successfully proved.

□

Theorem 4.3.11 (VI.11)**.** *To find a third straight line proportional to two given straight-lines. Let BA and AC be the two given straight lines, and let them be laid down encompassing a random angle. It is required to find a third straight line proportional to BA and AC.*

Proof. Construction steps:

```
point D 30 30
point B 50 30
line db D B
online A D B
point F 35 45
```

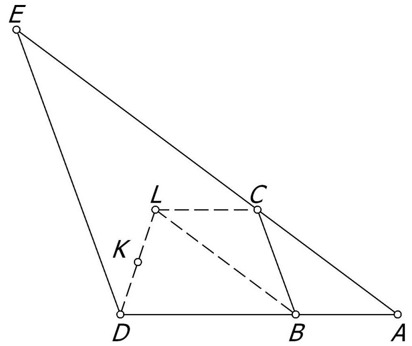


Figure 4.19: Theorem VI.11, construction at GCLC

```
line df D F
foot K B df
translate L D K K
translate C B A L
line bc B C
parallel de D bc
line ac A C
intersec E ac de
```

Comment on the construct. We cannot repeat Euclid's construction because we have neither compass nor segment transporter. We will do it differently for this. Let us first have AB and BD on one straight line. Let us construct the line segment $AC = BD$ on any angle. We choose the auxiliary point F to define the line *df*. On the line *df* we construct an isosceles triangle DLB (BK is altitude). Next, we construct the parallelogram $LBAC$. Now, we have $DB = LB = AC$. Through point D we will run a straight line *de* parallel to the segment CB. The intersection of the lines *de* and *ac* is the point E, and the segment CE the third straight-line proportional. First, by the prover we will check the equality of the segments DB and AC:

```
prove { same_length A C B D }
```

We will also check if CE is really the segment we are looking for:

```
prove { equal { sratio A B B D } { sratio A C C E } }
```

The conjecture is successfully proved.

□

Theorem 4.3.12 (VI.12)**.** *To find a fourth straight line proportional to three given straight-lines. Let A, B, and C be the three given straight lines. It is required to find a fourth straight line proportional to A, B, and C.*

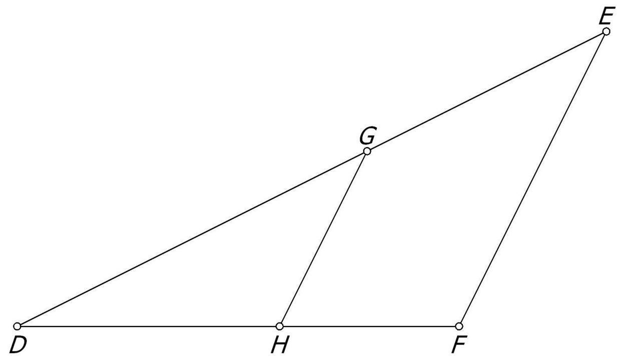


Figure 4.20: Theorem VI.12, construction at GCLC

Proof. Construction steps:

```
point D 30 30
point H 60 30
point G 70 50
online F H D
line gh G H
parallel ef F gh
line dg D G
intersec E dg ef
```

We completely repeat Euclid's construction and introduce the thesis for proof into the prover:

```
prove { equal { sratio D H H F } { sratio D G G E } }
```

The conjecture is successfully proved.

□

Theorem 4.3.13 (VI.13)**.** *To find the straight line in mean proportion to two given straight lines.*

Note: The construction of theorem VI.13 envisages the use of the circle and the intersection of the circle and the line. Prover GCLC has no tools to construct such things.

Theorem 4.3.14 (VI.14)**.** *Let AB and BC be equal and equiangular parallelograms having the angles at B equal. Let DB and BE be laid down straight-on with respect to one another. Thus, FB and BG are also straight-on with respect to one another. I say that the sides of AB and BC about the equal angles are reciprocally proportional, that is to say, that as DB is to BE, so GB is to BF.*

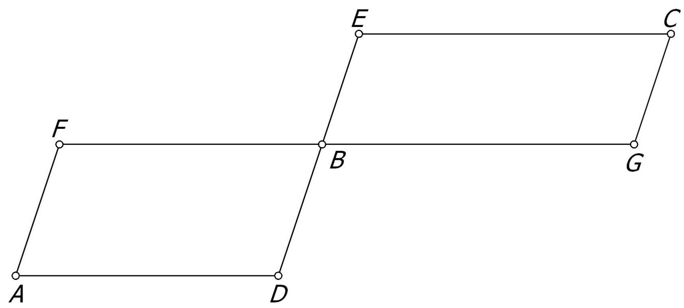


Figure 4.21: Theorem VI.14, construction at GCLC

Proof. Construction steps:

```
point A 10 20
point B 45 35
point D 40 20
online E D B
line db D B
line ad A D
translate F D A B
line fe F E
parallel dg D fe
line fb F B
intersec G fb dg
translate C B G E
```

We construct an $AFBD$ parallelogram using the translate command. We choose any point E on the line db. We look for the point G as the intersection of the line fb and the line parallel to the line ef through the point D. Next, we construct the parallelogram $BECG$.

We introduce the thesis for proof into the prover:

```
prove { equal { sratio G B B F } { sratio D B B E } }
```

The conjecture is successfully proved.

□

Theorem 4.3.15 (VI.15). *Let ABC and ADE be equal triangles having one angle equal to one angle, namely BAC equal to DAE. I say that in triangles ABC and ADE, the sides about the equal angles are reciprocally proportional, that is to say, that as CA is to AD, so EA is to AB.*

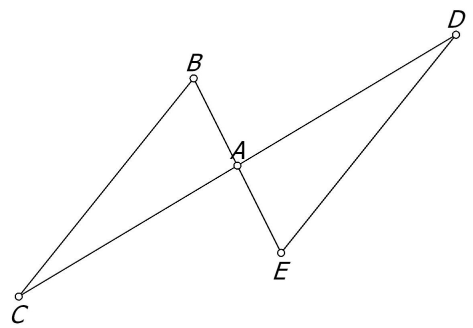


Figure 4.22: Theorem VI.15, construction at GCLC

Proof. Construction steps:

```
point A 45 30
point B 40 40
point C 20 15
translate E B A A
translate D C A A
```

We completely repeat Euclid's construction and introduce the thesis for proof into the prover:

```
prove { equal { sratio D H H F } { sratio D G G E } }
```

The conjecture is successfully proved.

□

Theorem 4.3.16 (VI.16)**.** *Let AB, CD, AG, and KC be four proportional straight lines, such that as AB is to CD, so KC is to AG. I say that the rectangle contained by AB and AG is equal to the rectangle contained by CD and KC.*

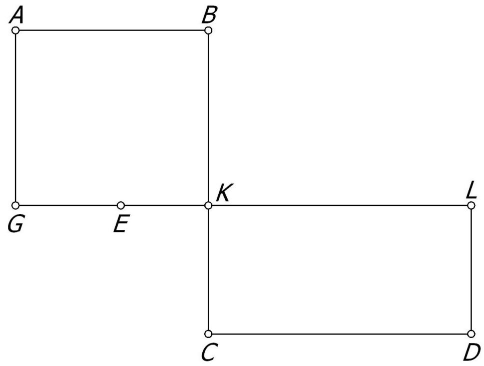


Figure 4.23: Theorem VI.16, construction at GCLC

Proof. Construction steps:

```
point E 20 20
point L 60 20
point B 30 40
line el E L
foot K B el
online G K L
line bl B L
parallel gc G bl
line bk B K
intersec C gc bk
translate A K G B
translate D K L C
```

We construct the rectangles so that GC is parallel to BL, so the condition $\frac{AB}{CD} = \frac{KC}{AG}$ is met. Next, we introduce the thesis for proof into the prover:

```
prove {equal { signed_area4 A B K G } { signed_area4 C K L D} }
```

The conjecture is successfully proved.

□

Theorem 4.3.17 (VI.17)**.** *Let A, B and C be three proportional straight lines, such that as A is to B, so B is to C. I say that the rectangle contained by A and C is equal to the square on B.*

Proof. Theorem VI.17 is a special case of theorem VI.16, when $AB = AG$. The prover proved theorem VI.16, that is, all cases are included and theorem VI.17 does not need a separate proof.

□

Theorem 4.3.18 (VI.18)**.** *Let AB, CD, AG, and KC be four proportional straight lines, such that as AB is to CD, so KC is to AG. I say that the rectangle contained by AB and AG is equal to the rectangle contained by CD and KC.*

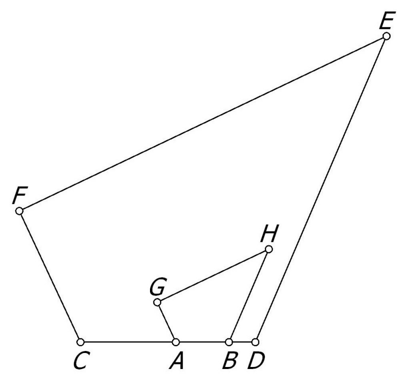


Figure 4.24: Theorem VI.18, construction at GCLC

Proof. Construction steps:

```
point C 10 10
point D 30 10
point F 3 25
point E 45 45
line cd C D
online A C D
```

```
online B C D
line ce C E
parallel ah A ce
line ed E D
parallel bh B ed
intersec H ah bh
line df D F
parallel bg B df
line fc F C
parallel ga A fc
intersec G ga bg
```

On the line *cd*, we choose any two points A and B. Than, by constructing appropriate parallel lines and their intersections, we construct a similar figure to the given. When checking the similarity, we have to check the proportion of sides three times:

```
prove { equal { sratio C D A B} { sratio D E B H } }

prove { equal { sratio F E G H }  { sratio C D A B} }

prove { equal  { sratio C D A B} { sratio F C G A } }
```

In each case, we get this result: The conjecture is successfully proved.

□

Theorem 4.3.19 (VI.19)**.** *Let ABC and DEF be similar triangles having the angle at B equal to the angle at E, and AB to BC, as DE is to EF, such that BC corresponds to EF. I say that triangle ABC has a squared ratio to triangle DEF with respect to that side BC has to EF.*

Proof. Construction steps:

```
point A 25 30
point B 20 20
```

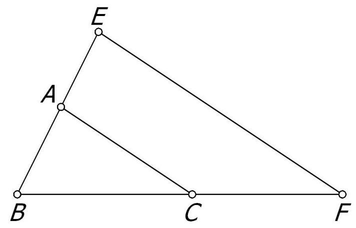


Figure 4.25: Theorem VI.19, construction at GCLC

```
point C 40 20
online E A B
cmark_t E
line ac A C
parallel ef E ac
line bc B C
intersec F ef bc
```

The construction has already been done many times and needs no description. Next, we introduce the thesis for proof into the prover:

```
prove { equal { ratio { signed_area3 A B C} { signed_area3 E B F } }
    { mult { sratio B C B F } { sratio B C B F} } }
```

This means the following in ordinary mathematical language:

$$\frac{S_{ABC}}{S_{EBF}} = \left(\frac{\overrightarrow{BC}}{\overrightarrow{BF}} \cdot \frac{\overrightarrow{BC}}{\overrightarrow{BF}}\right)$$

We get the result that the conjecture is successfully proved.

□

Chapter 5

Foundations of Geometry in Education

5.1 Criteria for textbooks review

In table 1.1, we apply concepts enabling one to characterize differences in presenting foundations of Euclidean geometry in classics such as (Fitzpatrick 2007, Hilbert 1899, Borsuk, Szmielew 1960, Tarski 1959, Hartshorne 2000). In this chapter, we apply them to review school textbooks. Here is a brief reminder and a rationale on how these criteria apply to school textbooks.

- Criterion 1: Primitive concepts. Whether there is a discussion concerning the concepts of point and line in a textbook. (We do not expect any analysis of primitive relations.)
- Criterion 2: The Pasch axiom. Whether there is a discussion in a textbook concerning the concept of a side of a line, a half-plane, the separation of a plane, or the cross-bar theorem. (We do not expect an explicit discussion of the Pasch axiom itself.)
- Criterion 3: Addition of line segments and angles, relation greater-lesser, subtracting a lesser from a greater; in short, algebra of line segments and angles. Whether a textbook introduces algebra of line segments and angles or simply processes their measures.
- Criterion 4: Absolute geometry and the parallel axiom. Whether criteria for congruent triangles precede the parallel axiom or a textbook puts them in reverse order. The pattern is, let us remind, that the absolute geometry contains criteria for congruent triangles.

Then, after introducing the parallel axiom, one has tools to prove some propositions concerning parallel lines, such as, e.g., *Elements*, I.33, 34.

5.2 School textbooks: absolute geometry and parallel lines

We choose the following Polish textbooks for our analysis (each one seeks to realize the so-called New Curriculum 2019):

(a) Kurczab, M., Kurczab, E., Świda, E.: *Podręcznik do liceów i techników. Zakres podstawowy. Klasa 1,* OE Pazdro, 2019. (Kurczab, Kurczab, Świda 2020)

(b) Babiański, W., Chańko, L., Janowicz, J., Ponczek, D., Wej, K.: *Matematyka 1. Podręcznik do matematyki dla liceum ogólnokształcącego i techniku. Zakres podstawowy i rozszerzony.*, Nowa Era, 2021. (Babiański, Chańko, Janowicz 2021)

(c) Antek, M., Belka, K., Grabowski, P.: *Prosto do matury 1. Matematyka. Podręcznik. Zakres podstawowy. Liceum i technikum*, Nowa Era, 2019. (Antek, Belka, Grabowski 2019)

(d) Dobrowolska, M., Karpiński, M., Lech, J.: *Matematyka z plusem 1. Podręcznik do liceum i technikum. Zakres podstawowy. Po szkole podstawowej*, GWO, 2019. (Dobrowolska, Karpiński, Lech 2019)

Primitive concepts are discussed only in the textbook (a); others do not touch on that topic

Now, let us focus on the fourth criterion. Textbook (a) cites Euclid's Postulates, including the celebrated Fifth. In the English translation, it reads: : *If a straight-line falling across two other straight-lines makes internal angles on the same side of itself whose sum is less than two right-angles, then the two other straight-lines, being produced to infinity, meet on that side of the original straight-line that the sum of the internal angles is less than two right-angles and do not meet on the other side.*

After introducing axioms and primitive concepts, in the next section, out of a sudden, the Fifth Postulate turns into the parallel postulate and takes another wording, namely: *Through a point not lying on a straight line, one and only one straight line can be drawn parallel to the given straight line.* This statement is a theorem rather than Euclid's or Hilbert's axiom.

Then, one finds supposed equivalent forms of the parallel postulate, i.a.:

1. *Angles in a triangle add up to* $180°$;

2. *Every triangle has a circumscribed circle*;

3. *There exists a square.*

In section 1.3.6, we showed that statements (1) and (3) are satisfied in the semi-Euclidean plane $\mathbb{L} \times \mathbb{L}$, meaning, they are not equivalent to the Fifth Postulate.

The theorems about congruent triangles follow after Thales' and Pythagoras' theorems. The theorems about the intersection of bisectors, medians, heights, and perpendicular bisectors in a triangle also precede criteria for congruent triangles. Due to such a sequence of feeding theorems, pupils have no tools to prove, for example, theorems about concurrent lines in a triangle.

In textbook (b), the chapter on plane geometry starts with the claim that angles in a triangle sum to π and its proof mirrors Euclid's I.32; authors seemingly assume that pupils already know parallel lines. Next, the textbook presents theorems about concurrent lines in a triangle and criteria for congruent triangles – all of them as simple truths with no proofs.

Textbooks (c) and (d) do not discuss issues related to parallel lines and congruent triangles as if these were facts get known by pupils in previous years.

Next, we will discuss modern Ukrainian textbooks:

(e) Мерзляк, А., Полонський, В., Якір, М.: *Геометрія: підручник для 7-го класу*, Гімназія, Харків 2015. (Мерзляк, Полонський, Якір 2015)

(f) Бевз, Г., Бевз, В., Владімірова, Н.: *Геометрія 7 клас*, Вежа, Київ 2015. (Бевз, Бевз, Владімірова 2015)

(g) Погорєлов, О.: *Геометрія 7-9 клас*, Школяр, Київ 2004. (Погорєлов 2004)

We should note that in Ukraine, starting with the seventh grade, the teaching of mathematics is divided into two parts: algebra and geometry. Therefore, a geometry course is quite comprehensive.

Now, textbooks (e) and (f) discuss primitive concepts. A point is said to be the simplest geometric figure. Textbook (e) calls a straight line – after Euclid – a perfectly even and infinite geometric figure; in textbook (f), one finds that a line is a figure with some geometric properties. They explain what is an axiom of a theory and discuss incidence axioms.

None of these textbooks discusses the half-plane, or side of the plane, or introduces the Pasch axiom.

Both textbooks, (e) and (f), assume that a line segment has length and introduce the concept of a unit segment. Similarly – that every angle has a measure. The textbook (e) defines equality of line segments and angles: Two line segments or angles are equal iff they can superimpose (*vide* Euclid's Common Notion 4). Addition and substruction of line segments are possible due to their measures.

After introducing basic concepts, textbook (e) examines equal triangles. The authors provide the so-called fundamental property of triangles which simply rewords Borsuk-Szmielew axiom C7. Euclid's technique of superimposing figures is applied to prove SAS and ASA criteria for congruent triangles. Due to the organization of the course, authors can show the properties of an isosceles triangle, the SSS condition, and theorems about parallel lines.

The textbook (f) introduces basic concepts and then turns to parallel lines. With no theorem about congruent triangles, the authors provide a strange proof of Euclid's I.27: it relies on the rotation and superposition of triangles. They also draw on the technique of superposing triangles when considering theorems on congruent triangles. Indeed, it is quite an unusual course.

Pogorelov's textbook (g) provides an axiomatic approach with nine axioms for plane geometry. Primitive concepts are points and straight lines denoted by blocks and small letters.

Instead of the Pasch axiom, Pogorelov introduces plane separation axiom: the straight line divides the plane into two half-planes (see Fig. 5.1).

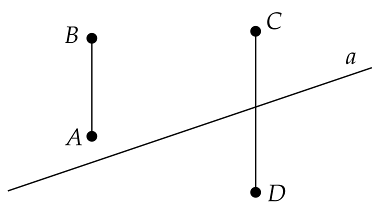


Figure 5.1: Half-planes by Pogorelov

Regarding axioms, instead of the Pasch axiom, Pogorelov introduces the plane separation axiom: the straight line divides the plane into two half-planes (see Fig. 5.1). On that basis, he proves a kind of the cross-bar theorem: *If a line that does not pass through any of the vertices of a triangle crosses one side of the triangle, then it crosses one and only one of the other two sides.*

Regarding algebra of line segments and angles, the textbook lists the following axioms:

- each segment has a length greater than zero. The length of the segment equals the sum of lengths of the parts, into which it is divided by any of its points;

- each angle has a measure greater than zero. The measure of the angle equals the sum of the measures of the angles into which it is divided by any of the angle's rays.

It means that the addition and subtraction of segments and angles result from the properties of real numbers.

Before introducing theorems about congruent triangles, Pogorelov adds an axiom about the existence of an equal triangle for each triangle on a given half-line in a given half-plane (vide Borsuk, Szmielew's system, axiom C7). Thus, Pogorelov's system clearly combines Hilbert's and Borsuk-Szmielew's axioms.

Finally, Pogorelov respects the classic arrangement between absolute geometry and the theory of parallel line: after introducing axioms, he spells out theorems about congruent triangles. Yet, he proves SAS, SSS, and ASA conditions by superimposed triangles.

Summary. Through the table 5.1, we review the discussed textbooks in terms of basic ideas developed by classics in synthetic geometry and tallied in Table 1.1. Its organization is as follows: + or - in a column means that a topic, e.g., the Pasch axiom, is or is not discussed in a textbook, *reordered* means that the parallel axiom precedes criteria for congruent triangles, while *classic* – that the parallel axiom follows after criteria for congruent triangles. Finally, that algebra of line segments and angles, addition, subtraction (a lesser from the greater), and the relationships greater-lesser, is introduced by real numbers means that every textbook discerns a line segment or an angle, say, AB, α, and its measure $|AB|$, $\measuredangle\alpha$. Each textbook applies algebra to measures and does not clarify the difference between a line segment or angle and its measure. Moreover, usually, algebra applies to measures of triangles defined as half the product of its base and height. Consequently, the Thales' theorem, even though rephrased in the wording of proportions and ratios, literally means equality of two quotients, say $\frac{|\triangle AED|}{|\triangle EDB|} = \frac{|AD|}{|DB|}$ (see Fig.

3.3). Out of the classics in synthetic geometry, only [Borsuk, Szmielew 1972] introduces the measure of a triangle.

Textbook	**Primitive concepts**	**the Pasch axiom**	**Algebra of line segments and angles**	**Absolute geometry + the parallel axiom**
(a)	+	–	by real numbers	reordered
(b)	–	–	by real numbers	reordered
(c)	–	–	by real numbers	-
(d)	–	–	by real numbers	-
(e)	+	in terms of half-plane	by real numbers	classic
(f)	+	–	by real numbers	reordered
(g)	+	in terms of half-plane, the Pasch Axiom is a theorem	by real numbers	classic

Table 5.1: Comparison of school textbooks

5.3 University Courses and Textbooks: absolute geometry and parallel lines

Below we go through syllabuses on elementary geometry in Polish universities.

1. University of Gdańsk, Mathematics, Elementary geometry. The course starts with the theorems about triangles, the syllabus does not mention the axiomatic method. `https://mat.ug.edu.pl/wp-content/uploads/2020/02/SYL_Geometria_elementarna_2020-1569362079709.pdf`

2. University of Zielona Góra, Mathematics, Elementary Geometry. The course starts with an axiomatic method in geometry. Presents axioms of Euclidean geometry, including various forms of the parallel postulate. Recommended textbook: R. Doman, *Lectures on elementary geometry.* `https://webapps.uz.zgora.pl/syl/index.php?/course/showCourseDetails 995106`

3. Jagiellonian University in Krakow, Mathematics (for teachers), Geometry 1. The course tarts with the basic theorems of elementary geometry; the syllabus does not mention the axiomatic method. Recommended textbook: R. Doman, *Lectures on elementary geometry* `https://sylabus.uj.edu.pl/pl/document/c87c14cc-d2d4-4ad3-8bd3-ebdd1a79d62e.pdf`

4. Nicolaus Copernicus University in Toruń, Mathematics and Physics, Mathematics and Informatics, Geometry. The course studies ways of defining geometry examplified by affine geometry; introduces an axiomatic method and the idea of a model of an axiomatic system. `https://usosweb.umk.pl/kontroler.php?\_action=katalog2/przedmioty/pokazPrzedmiot\&kod=1000-M1GEOz`

5. The Pedagogical University of Krakow, Mathematics, Elementary Geometry. The course provides mere information on Hilbert's axioms and develops geometry in the Cartesian plane $\mathbb{R} \times \mathbb{R}$. Recommended textbook: J. Szczawińska, J Szpond. *Elementary Geometry.* Omega, 2018 `https://matematyka.up.krakow.pl/pliki/planprognew/I/2021\%7C22/stac/karty/Karty-kursow-przedmiotow-planu-glownego.pdf#page=35&zoom=100,90,94`

6. University of Rzeszow, Mathematics, School Geometry. The course starts with a brief history of geometry and in that context mentions Euclid's *Elements*. Recommended textbooks: H. Coxeter, *Introduction to old and new geometry*, and R. Doman, *Lectures on elementary geometry.* `https://www.ur.edu.pl/storage/file/core_files/2021/11/23/9419c68abe718dd9d0dac6b4807cf13a/Geometria\%20szkolna.pdf`

Doman's *Lectures on elementary geometry* (Doman 2001), is the most often recommended textbook throughout syllabuses. Chapter 8 of that book, *Axiomatic Method in Geometry*, contains a stylization of Hilbert's system: instead of congruence axioms, it introduces axioms for the metric plane. Indeed, the textbook does not present any typical synthetic geometry arguments.

Elementary Geometry (Szczawińska, Szpond 2018) in the first chapter introduces Hilbert's axioms and provides an example of arguments based on axioms by proving the theorem: Every line segment has a middle point. While the classic proof proceeds in the absolute geometry, that sample proof relies on the properties of a parallelogram.

Overview of syllabuses for elementary geometry (or related) courses in Ukrainian universities.

1. Yuriy Fedkovych National University of Chernivtsi, Teachers' Mathematics, Foundations of Geometry. `http://ztimathan.chnu.edu.ua/wp-content/uploads/2021/11/24_Base_geom.pdf`

   The course aims to ensure a thorough mastering of Hilbert's axiomatics; promotes the formation of skills in the application of theoretical knowledge to the proofs of theorems and the implementation of axioms of Euclidean geometry in the Cartesian plane. Sample topics: Topic 1. Introduction - historical information. Topic 2. Hilbert's axioms Topic 3. Other versions of Hilbert's axioms. (16 hours)

2. Uzhhorod National University, Teachers' Mathematics, Foundations of Geometry. `https://www.uzhnu.edu.ua/uk/infocentre/get/39973`

   Module 1. Euclidean geometry: Topic 1. A brief history of geometry. Topic 2. Hilbert's axioms of Euclidean geometry. Topic 3. Weyl's axioms of Euclidean geometry. Topic 4. Modern axioms of Euclidean geometry. Topic 5. Axioms for a school course on geometry. An overview of various approaches to a school geometry course: Atanasyan's axioms, Aleksandrov's axioms, Pogorelov's axioms.

   Module 2. Non-Euclidean geometry: Topic 1. Elements of Lobachevsky's geometry. Topic 2. Elements of Riemann geometry.

3. B. Hrinchenko University of Kyiv, Teachers' Mathematics, Foundations of Geometry.

   Module 1. Axioms of geometry. Topic 1. Axioms of Euclidean geometry. Topic 2. Axioms of incidence and order. Topic 3. Axioms of congruence and continuity. Module

II. Axiomatic systems of Euclidean and non-Euclidean geometries: Topic 4. The Fifth Postulate of Euclid. Topic 5. Lobachevsky's axiom of parallelism. Topic 6. Models of Lobachevsky's geometry.

4. Ivan Franko National University of Lviv, Teachers' Mathematic, History of Mathematics.

   Contains following topics: Predecessors and creators of analysis infinitesimals; creators of non-Euclidean geometries; two paradigms for constructing mathematical theories (Euclid's Principles, Hilbert's Geometry).

5.4 Thales' theorem in school and university textbooks

All the reviewed Polish school textbooks present the proof of Thales' theorem in terms of the areas of triangles and lengths of line segments. They emulate VI.2 by substituting a triangle with its area and a line segment with its length. Some textbooks introduce Euclid's Theorem VI.1 before proving Thales' theorem.

In Ukrainian textbooks, Thales' theorem gets the following wording: *If the parallel lines intersect sides of an angle and cut off equal segments on one side, then they also cut off equal segments on the other side.*

A counterpart of Euclid's VI.2 is called the theorem on proportional segments. Merzlak (Мерзляк, Полонський, Якір 2015), seeking to prove it, first considers the rational proportion case. The general case proceeds in *reductio ad absurdum* mode and mirrors Milman-Parker's proof. The only difference is that instead of showing the equality of ratios, he considers inequality and in this way he reaches a contradiction. Due to this trick, he can omit the concept of the limit of a sequence. Pogorelov's textbook (Погорєлов 2004) adopts the same approach. Since the proof is quite involved, Pogorelov recommends it be not obligatory.

Bevz's textbook presents the theorem on proportional segments without proof.

Regarding academic textbooks, Doman's (Doman 2001) does not discuss Thales' theorem, although it refers to it in the proofs of other theorems.

Chapter 3 of Szczawińska, Szpond's (Szczawińska, Szpond 2018) is dedicated to Thales' theorem. Firstly, the authors introduce the Euclidean norm, a ratio of division, and a determinant. Thales' theorem is rephrased in terms of the ratio of division as follows: *Let P, Q and*

R be non-collinear points and points Q' and R' lie on the lines pr PQ and pr PR, respectively, and both P, Q, Q' and P, R, R' are pairwise different a) the lines pr QR and pr $Q'R'$ are parallel b) the partition ratios of $S(P, R; R')$ and $S(P, Q; Q')$ are equal.

Proof of Thales' theorem builds on the properties of the division ratio and a determinant. In (Petiurenko 2021), we interpret that proof in terms of the area method showing that the determinant as applied in the proof is a signed area of a triangle on the Cartesian plane $\mathbb{R} \times \mathbb{R}$.

Chapter 6

Conclusions

Throughout the thesis, we have already made some recommendations concerning the secondary school curriculum and university courses on synthetic geometry. Over this section, we put them together.

6.1 Recommendations for University Courses

(1) The course Foundation of Geometry for prospective teachers should begin with the modernized Hilbert's axioms (Section 1.1.1).

(2) Then students compare Euclid's I.4 and Hilbert's axiom C6 (Side-Angle-Side criterion for congruent triangles). They study all theorems necessary for proving theorems on congruent triangles. The course proceeds within absolute geometry (Section 1.1.2).

(3) Students prove theorems about congruent triangles, namely: Side-Side-Side, Angle-Side-Angle, Angle-Angle-Side, Side-Side-90°. (Axiom C6, Theorems 1.1.3, 1.1.4, 1.1.10, 1.1.12)

(4) Students prove theorems about concurrent lines in a triangle. Students examine which of them do not require the parallel postulate. (Theorems 1.1.13, 1.1.14, 1.1.15, 1.1.20).

(5) Hilbert's and Euclid's systems are compared (Section 1.2)

(6) Students get to know Borsuk-Szmielew's and Tarski's axioms and compare four classic systems of Euclidean geometry in terms of primitive concepts, the Pasch axiom, the SAS axiom (Sections 1.4).

(7) Non-Euclidean and semi-Euclidean plane (Section 1.5.4). Students show the indepen-

dence of the Fifth Postulate. They show Euclid's propositions which require the Fifth Postulate (Section 1.5.5).

The main goal of the course is to provide teachers with tools, enabling them to analyze various axiomatic systems of geometry. The rationale for that course is this: school textbooks merge axiomatic systems, and pupils get information from a variety of sources. A teacher should be able to figure out an axiomatic background of a specific proof he deals with through the teaching process..

6.2 Recommendations for secondary schools concerning absolute geometry and the parallel axiom

First of all, instead of pondering on primitive concepts, we recommend introducing Euclidean geometry through the app *eculidea* (https://www.euclidea.xyz/).

The application *euclidea* provides a IT-game-style introduction to straightedge and compass constructions. It includes 148 construction tasks grouped in fifteen books named with Greek alphabet letters. Starting with Postulates 1 and 3, the construction of the equilateral triangle makes the first task. Then it introduces new constructions tools, such as perpendicular bisector (α, 2), midpoint of a line segment (α, 3), perpendicular to a line through a point (β, 6, 7), etc. As the game proceeds, these constructions make new tools on a par with straightedge and compass.

The parallel line through a point is drawn as a perpendicular to the perpendicular (ε 1); parallel transportation of an angle and a line segment (ε, 5, ε, 6) is accomplished by the means of the parallel line. Transportation of an angle or a line segment to a point is not considered at all. Clearly, despite stylizations, *euclidea* in no way echoes the deductive structure of the *Elements*.

(1) The concept of point, straight line, angle, circle and straightedge, and compass construction tools, we introduce through the following tasks of *euclidea*, book α:

- Construction of an equilateral triangle (Section 1.2.1 Fig. 1.22)
- Transportation of the line segment (Section 1.2.1 Fig. 1.23, 1.24)

- Construction of the bisector of the angle (Section 1.2.4 Fig. 1.32)
- Cut a given the segment in half (Section 1.2.4 Fig. 1.33)
- Construction of the perpendicular to the straight line (Section 1.2.4 Fig. 1.35)
- Construction of a triangle from three sides (transportation of the angle) (Section 1.2.9 Fig. 1.49)

(2) We assume that the segments have lengths and angles are measured. We get the addition and subtraction of segments and angles and the relation greater-lesser from the properties of real numbers. In short, we do not problematize the algebra of segments and angles. However, teachers are aware of this.

(3) The theorems about congruent triangles are given without proofs: Side-Angle-Side, Side-Side-Side, Angle-Side-Angle, Angle-Angle-Side, Side-Side-90°. With these tools, pupils can justify constructions introduced in point (1).

4) We present all theorems about concurrent lines in a triangle as starting point theorems. As for proofs, one can provide tips on using specific congruence criteria. (Theorems 1.1.13, 1.1.14, 1.1.15, 1.1.20).

(6) The theory of parallel lines:

Theorems on two straight lines intersecting a third (with hints to proofs) (Section 1.2.11 Fig. 1.58, 1.59, 1.60).

Theorem about sum of angles in a triangle (with hints to proofs) (Section 1.2.12 Fig. 1.64).

Parallelogram theorems (with hints to proofs) (Section 1.2.13 Fig. 1.65, 1.66).

6.3 Recommendations concerning Thales' theorem

Recommendations for school curriculum:

(1) Detailed discussion of Theorem VI.1,

(2) Solving geometric problems using theorem VI.1,

(3) Introducing the co-side theorem (as a generalization of theorem VI.1),

(4) Solving geometric problems using the co-side theorem using four diagrammatic patterns (Sections 3.4.2, 3.4.3).

Recommendations for university courses:

(1) Discussion of theorem V1.1.

(2) The co-side theorem and various methods of its proof.

(4) Four schemes for solving geometric problems using the co-side theorem.

(5) Students gather tasks that come down to the co-side theorem.

(6) Introducing the lemmas of elimination.

(7) Schema of the proof of theorems by the method of elimination of points.

(8) Automatic theorem proving in GCLC:

a) Program interface

b) Constructions allowed in GCLC-prover

c) Formulating the thesis of proof in terms of GCLC.

Chapter 4 details all these topics.

Bibliography

Antek, M., Belka, K., Grabowski, P.: *Prosto do matury 1. Matematyka. Podręcznik. Zakres podstawowy. Liceum i technikum*, Nowa Era, 2019.

Avigad, J., Dean, E., Mumma, J.: A Formal System for Euclid's Elements. *The Review of Symbolic Logic* 2(4), 700–768, 2009.

Babiański, W., Chańko, L., Czarnowska, J., Mojsiewicz, B., Wesołowska, J.: *Teraz Matura. Matematyka. Poziom rozszerzony. Zbiór zadań i zestawów maturalnych*, Nowa Era, Warszawa 2019.

Babiański, W., Chańko, L., Janowicz, J., Ponczek, D., Wej, K.: *Matematyka 1. Podręcznik do matematyki dla liceum ogólnokształcącego i techniku. Zakres podstawowy i rozszerzony.*, Nowa Era, 2021.

Beeson, M.: On the Notion of Equal Figures in Euclid; `https://www.researchgate.net/publication/343986568` [visited on 16 June, 2022]

Beeson, M.: Constructive geometry. In: Arai, T. (ed.) et al., *Proceedings of the 10th Asian logic conference, Kobe, Japan, September 1–6, 2008*. Hackensack, NJ, World Scientific, 19-84, 2010; `https://www.researchgate.net/publication/264957027` [visited on 16 June, 2022]

Beeson, M., Narboux, J., Wiedijk, F.: Proof-checking Euclid. *Annals of Mathematics and Artificial Intelligence* 85, 213–257, 2019.

Birkhoff, G.: A Set of Postulates for Plane Geometry, Based on Scale and Protractor. *Annales of Mathematics* 33(2), 329–345, 1932.

Błaszczyk, P.: Descartes' Transformation of Greek Notion of Proportionality. In B. Sriraman (ed.) *Handbook of the History and Philosophy of Mathematical Practice*. Springer, Cham, 2021; https://link.springer.com/referenceworkentry/10.1007/978-3-030-19071-2_16-1

Błaszczyk, P.: Galileo's paradox and numerosities. *Zagadnienia Filozoficzne w Nauce* 70, 73–107, 2021.

Błaszczyk, P.: From Euclid's *Elements* to the methodology of mathematics. Two ways of viewing mathematical theory. *AUPC* 10, 5–15, 2018; https://didacticammath.up.krakow.pl/article/view/6613

Błaszczyk, P.: A Purely Algebraic Proof of the Fundamental Theorem of Algebra. *AUPC* 8, 6–22, 2016; https://didacticammath.up.krakow.pl/article/view/3638

Błaszczyk, P., Major, J.: Calculus without the Concept of Limit. *AUPC* 6, 19–38, 2014; https://didacticammath.up.krakow.pl/article/view/3654

Błaszczyk, P. Mrówka, K.: *Euklides, Elementy, Księgi V–VI. Tłumaczenie i komentarz*. Copernicus Center Press, Kraków 2013.

Błaszczyk, P. Mrówka, K.: *Kartezjusz, Geometria. Tłumaczenie i komentarz*. Universitas, Kraków 2015.

Błaszczyk, P. Mrówka, K., Petiurenko, A.: Decoding Book II of the *Elements*. *AUPC* 12, 39–88, 2020; https://didacticammath.up.krakow.pl/article/view/8462

Błaszczyk, P., Petiurenko, A.: Commentary to Book I of the *Elements*. *AUPC* 13, 43–93, 2021; https://didacticammath.up.krakow.pl/index.php/aupcsdmp/article/view/6901

Błaszczyk, P., Petiurenko, A.: Euclid's proportion revised. *AUPC* 11, 37–61, 2019; https://didacticammath.up.krakow.pl/index.php/aupcsdmp/article/view/6901

Błaszczyk, P., Petiurenko, A.: Euler's Series for Sine and Cosine. An Interpretation in Nonstandard Analysis. In: Waszek, D., Zack, M. (eds.). *Annals of the CSHPM*. Birkhauser 2022 (to appear).

Błaszczyk, P., Petiurenko, A.: On diagrams accompanying *reductio ad absurdum* proofs in Euclid's *Elements* book I. Reviewing Hartshorne and Manders; `https://arxiv.org/pdf/2206.12213.pdf`; `https://www.researchgate.net/publication/361559448`

Borsuk, K., Szmielew, W.: *Foundations of Geometry*, North-Holland, Amsterdam 1960.

Borsuk, K., Szmielew, W.: *Podstawy Geometrii*, PWN, Warszawa 1972.

Boutry, P.: *On the Formalization of Foundations of Geometry.* Logic in Computer Science [cs.LO] Université de Strasbourg, 2018.

Chou, S., Gao, X., Zhang, J.: *Machine Proofs in Geometry*, World Scientific, Singapore 1994.

Dehn, M.: Legendre'schen Sätze über die Winkelsumme im Dreieck. *Mathematische Annalen* 53(3), 404–439, 1900.

De Risi, V.: Euclid's Common Notions and the Theory of Equivalence. *Foundations of Science* 26, 301–324, 2021.

De Risi, V.: The development of Euclidean axiomatics The systems of principles and the foundations of mathematics in editions of the Elements in the Early Modern Age. *Archive for the History of Exact Science* 70, 591– 676, 2016.

Dobrowolska, M., Karpiński, M., Lech, J.: *Matematyka z plusem 1. Podręcznik do liceum i technikum. Zakres podstawowy. Po szkole podstawowej*, GWO, 2019.

Doman, R.: *Wykłady z geometrii elementarnej*, Wydawnictwo Naukowe UAM, Poznań 2001.

`https://www.euclidea.xyz/`

Fitzpatrick, R.: *Euclid's Elements of Geometry translated by R. Fiztpatrick*, 2007; http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

GCLC. `http://poincare.matf.bg.ac.rs/~janicic//gclc/`

GCLC manual. `http://poincare.matf.bg.ac.rs/~janicic//gclc/gclc_man.pdf`

Greenberg, M.: *Euclidean and non-Euclidean Geometries.* Freeman, New York 2008.

Hartshorne, R.: Geometry: Euclid and Beyond. Springer, New York 2000.

Hartshorne, R.: Teaching Geometry According to Euclid. *Notices of the American Mathematical Society* 47, 460–465, 2000.

Hilbert, D.: Grundlagen der Geometrie. Festschrift Zur Feier Der Enthüllung Des Gauss-Weber-Denkmals in Göttingen. Teubner, Leipzig 1899, 1–92. In: K. Volkert (Hrsg.), David Hilbert,*Grundalgen der Geometrie* (Festschrift 1899), Springer, Berlin 2015.

Hilbert D.: *Grundlagen der Geometrie.* 11. Auflage. Stuttgart 1972.

Hilbert, D.: Über den Zahlbegriff. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 8, 180–184, 1900.

Janičić,P., Narboux, J., Quaresma, P.: *The area method: a Recapitulation*, Journal of Automated Reasoning, Springer Verlag 48(4), 489–532, 2012.

Kurczab, M., Kurczab, E., Świda, E.: *Podręcznik do liceów i techników. Zakres podstawowy. Klasa 1,* OE Pazdro, 2019.

Kurczab, M., Kurczab, E.: *Podręcznik do liceów i techników. Zakres podstawowy. Klasa 2,* OE Pazdro, 2020.

Martin, G.: *Geometric Constructions.* Springer, Berlin 1998.

Millman, R., Parker, G.: *Geometry A Metric Approach with Models*, Springer, Berlin 1991.

Mueller, I.: *Philosophy of Mathematics and Deductive Structure in Euclid's Elements.* Dover, New York 2006 (first edition: MIT Press, Cambridge, Massachusetts 1981).

Pambuccian, V.: Schacht, C. The Ubiquitous Axiom. *Results in Mathematics* 76, 114, 2021.

Pambuccian, V.: Zur Existenz Gleichseitiger Dreiecke in H-Ebenen, *Journal of Geometry* 63(1–2), 147–153, 1998.

Petiurenko A.: Solving Math Problems Using the Area Method, *Proceedings of the conference Contemporary Mathematics in Kielce 2020*, February 24-27, Sciendo, 2021, 233–246.

Petiurenko, A.: Wizualizacja wyznacznika w metodzie pola. In: Formicki, G., Szmańda, J. (eds.)*Znaczenie badan scisłych i przyrodniczych prowadzonych w Uniwersytecie Pedagogicznym im. KEN w Krakowie w rozwoju nauki*, Wydawnictwo Naukowe Uniwersytetu Pepagogicznego w Krakowie, 2021, 119–130. `https://www.researchgate.net/publication/361727817`

Pollard S.(ed.): *Essays on the Foundations of Mathematics by Moritz Pasch.* Springer, New York 2010.

Quaresma, P., Janičić, P.: Framework for Constructive Geometry (Based on Area Method)(Version 1.20). `http://poincare.matf.bg.ac.rs/~janicic//gclc/`

Stankowa, Z., Rike T.:*A Decade of the Berkeley Math Circle: The American Experience*, American Mathematical Soc. 2008.

Szczawińska, J., Szpond, J.: *Geometria elementarna. Notatki do wykładu*, wydanie II zmienione. Wydawnictwo Szkolne Omega, Kraków 2018.

Schwabhäuser, W., Szmielew, W., Tarski, A.: *Metamathematische Methoden in der Geometrie,* Springer, Berlin 1983.

Tarski, A.: What is elementary geometry? In: Henkin, L., Suppes, P., Tarski, A. (eds.) *The Axiomatic Method*, North-Holland, Amsterdam 1959, 16–29.

Tarski, A., Givant, S.: Tarski's System of Geometry. *Bulletin of Symbolic Logic* 5, 175–214, 1999.

Бевз, Г., Бевз, В., Владімірова, Н.: *Геометрія 7 клас*, Вежа, Київ 2015.

Мерзляк, А., Полонський, В., Якір, М.: *Геометрія: підручник для 7-го класу*, Гімназія, Харків 2015.

Погорєлов, О.: *Геометрія 7-9 клас*, Школяр, Київ 2004.