Pedagogical University of Cracow
Faculty of Exact and Natural Sciences
Department of Mathematics

Review of the Thesis „Foundations of geometry for secondary schools and prospective teachers“ by Anna Petiurenko submitted for the degree of Doctor of Philosophy

I was appointed by the Scientific Council for Mathematics of the Pedagogical University of Cracow to be one of reviewers of the doctoral dissertation „Foundations of geometry for secondary schools and prospective teachers“ submitted by Anna Petiurenko for the degree of Doctor of Philosophy at the forementioned university (Pedagogical University of Cracow, Faculty of Exact and Natural Sciences, Department of Mathematics).

The thesis belongs to the intersection of the following mathematical fields: mathematical education, mathematical logic, geometry, and mathematical software (which are also my areas of research).

The following text gives a brief overview of the thesis, its main contributions, and gives a final opinion on its valuae.

1 Introduction

Foundations and teaching of geometry have been an extremely important topic through the history of mathematical. For instance, formal development of geometry had a central role in establishing the deductive method in Ancient Greece and in building the modern mathematics at the turn of the 20th century. In modern times, IT technologies bring new opportunities and challenges for geometry: geometry has one of central roles in computer-based theorem proving, from the very beginning of this field in early 1950's. Modern automated theorem provers can prove many geometry theorems, but they still need proper approaches in order to be used in mathematical education. Hundreds of researchers contributed to development of different geometry systems, including those suitable for computer-based theorem proving. In particular, there are many significant contributors in this fields that originate from Poland, to name just Alfred Tarski, Karol Borsuk, and Wanda Szmielew, which gives a certain tone to this thesis submitted to Pedagogical University of Cracow.

2 Structure and Contents of the Thesis

The thesis has 179 pages and around 60 bibliographic references. The thesis consists of the introduction and six chapters:

Introduction which briefly lists and explains the contents of the thesis.

Chapter 1: Plane geometry which explains building Euclidean geometry using Hartshorne's axiom system. Classical, key theorems of this theory, including theorems ensuring congruence of triangles and Pytagorean theorem, are proved in details. A deep analysis and reconstruction of Euclid's proofs from the Elements are given. Construction of auxiliary points needed in some proofs are effectively constructed using the tools straightedge and compass. Some digressions about space geometry are also given (e.g., Theorem 1.1.22 and Theorem 1.1.23 in Section 1.1.5), although the axioms for space geometry are not listed. There are thorough discussions on relations ,,equal to", ,,congruent", ,,lesser-than", ,,commensurable", but also on some more exotic such as "line standing on another line". Models of Euclidean geometry and it variations are considered (for instance, there is an analysis of Euclid's propositions that do not hold in $\mathbf{L} \times \mathbf{L}$).

Chapter 2: Thales' Theorem through the 20th-century Foundations of Geometry discusses how Hilbert, Hartshorne and Szmielew-Tarski develop arithmetic of line segments and theory of proportions. Also, it is explained how Borsuk-Szmielew's system, Birkhoff's system, and Millman-Parker's system introduce measure of line segments and how Thales' theorem can be expressed in terms of measures of line segments.

Chapter 3: The Area Method and Thales's Theorem gives a background of the area method and its axiom system. The area method is a decision procedure for one fragment of Euclidean plane geometry and is suitable for automation. The Euclid's proof of Thales' theorem and its counterpart based on the area method are presented and discussed. A model of the area method axioms in Cartesian plane is built, ensuring consistency of the axioms. Theorems crucial for the method application (primarily the co-side theorem) were proved using the area method axioms. A range of of beautiful geometry problems, both elementary and advanced, were proved using the co-side theorem.

Chapter 4: Automatic Theorem Proving Based on the Area Method lists all the elimination lemmas used by the area method, some of which with proofs. A detailed presentation of the area method is given. Several examples, proved by the automated theorem prover built into the tool GCLC are given and explained in details. There is a number of interesting observations about limitations that the area method and the prover GCLC have (e.g., conjectures involving angle bisectors or intersections of the circle and the line cannot be handled). The prover GCLC was applied to theorems from Euclid's Book VI and it was shown that it can prove most of them.

Chapter 5: Foundations of Geometry in Education which provides analysis and comparison, concerning several basic issues (such as primitive concepts or parallel axioms), of a number of high-school textbooks used in Poland and in Ukraine. Similar analysis is presented for curricula and textbooks of a number of universities in Poland and in Ukraine.

Chapter 6: Conclusion which, following analyses presented in the previous chapters, provides recommendations for university and secondary schools courses in synthetic geometry: what axioms systems should be taught and compared, what primitive notions are to be used, what mathematical software should be used, in which way, etc.

3 Analysis of the Thesis and its Main Results

The thesis belongs to the intersection of several mathematical fields, primarily mathematical education, geometry, mathematical logic, and mathematical software. Foundations and teaching of geometry have been an extremely important topic throughout the history of mathematics and mathematical education, while modern IT technologies bring new opportunities and challenges. This thesis addresses mathematical results spanning a time period of around 2500 years: from ancient Greece to modern IT technologies. The thesis deeply analyses several axiom systems, primarily those due to Euclid (≈300 BC), Hilbert (1899), Tarski (1959), Borsuk/Szmielew (1960), Chou/Gao/Zhang (1994), Hartshorne (2000). The analysis shows that there is still no "the best" approach for developing and teaching geometry, but brings us to some recommendations on how a modern school and university courses in geometry should look like.

This thesis brings a number of contributions. Among the central ones are analysis and comparison of a number of axiom systems for geometry, especially in terms of means for proving Thales' theorem. This theorem is important in many aspects, one of which is that it pave ways for automated proving of geometry theorems. In particular, the thesis discusses the role of Thales' theorem in the area method – one of the most important approaches for automated theorem proving in geometry. The thesis brings a construction of a model of semi-Euclidean plane (a plane in which the sum of angles in a triangle equals π, but the parallel postulate does not hold). Also, a model of the axioms of the area method is constructed in the Cartesian plane. The thesis links some ancient, and almost forgotten concepts confronts with its modern counterparts or reformulations. Especially interesting are proofs of theorems from Euclid's Book VI constructed automatically. An important output of this thesis are also recommendations on how a modern school and university courses in geometry should look like.

Apart from the original contributions, the thesis also provides a very good overview of the relevant literature. The bibliography is detailed, covering many relevant sources, from classical one to recent ones.

The thesis is very well written, it is both very readable and precise.

4 Conclusions

I find that, by this thesis, Anna Petiurenko has shown a high degree of research maturity that links several mathematical fields, including mathematical education, geometry, mathematical logic, and mathematical software. She has shown that she can acquire and systematize knowledge from one research field, to critically analyse it, and bring new contributions. Anna Petiurenko reached a number of significant contributions and conclusions about different geometry system and provided important recommendations for future school and university courses in geometry. Some results form the thesis have been published in research papers in reputable international journals.

Following all of the above, I find that the thesis satisfies requirements for a PhD degree in mathematics, and hence **I recommend that the proposed thesis is accepted.**

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