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Referee's report on the manuscript

"Foundations of geometry for secondary schools and prospective teachers"
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submitted as doctoral thesis to Pedagogical University of Cracow, Faculty of Exact and Natural Sciences, Department of Mathematics

Key words: absolute geometry, Euclidean geometry, Euclid's geometry, Cartesian plane over hyperreals, semi-Euclidean plane, axioms for geometry, Euclid, Hilbert, Hartshorne, Borsuk, Szmielew, Tarski, Thales' theorem, area method, co-side theorem, automated theorem proving, proportion, real numbers, *Elements* Book VI, secondary school curriculum, prospective teachers of mathematics

Introduction

Axiomatic foundations of plane geometry were studied through centuries, ever since the times of Euclid (4th century BC), playing the pivotal role in the development of mathematical rigor in teaching and understanding mathematics. Some of the prominent mathematical minds in the 19th and 20th century continued to study fundamentals of Euclidean geometry: Hilbert, Tarsky, Hartshorne are some famous names in this list. As a result, today we know much more about interconnections which do exist in this theory, concerning the axiom of parallels, the sum of angles theorem, the notion of similarity, the notions of length and area.

From the times of Ancient Greek and Euclid's ΣΤΟΙΧΕΙΑ – *Elements*, synthetic geometry of the plane played a fundamental role in education. In contemporary education process it is a vital and inevitable part of school curricula. However, in the last century there has been a lot of debate about the way this topic should be delivered to pupils on one hand, and the level of knowledge of the subject which should be taught to prospective school teachers on the other. The work of the author sheds some new light and gives new insight into both of these problems.

The manuscript

The manuscript is typeset in TeX and has 178 pages, printed double-sided on A4 paper with proper font size, line spacing and margins. It consists of the following parts. After *Introduction* (pp. 4-10), there are Chapter 1. *Plane geometry* (pp. 11-83), Chapter 2. *Thales' theorem through the 20^th^ century foundations of geometry* (pp. 84-102), Chapter 3. *The area method and Thales' theorem* (pp. 103-126), Chapter 4. *Automatic theorem proving based on the area method* (pp. 127-159), Chapter 5. *Foundations of geometry in education* (pp. 160-169) and Chapter 6. *Conclusions* (pp. 170-173). The list of references is on four pages 174-178.

The first chapter, the longest one in the manuscript, on 73 pages contains detailed analysis of current axiomatics of plane geometry. The author presents the axiomatics of plane geometry following Hartshorne's exposition of Hilbert's axioms. Section 1.2 (pp. 28-66) contains complete review of all theorems of Book I of Euclid's *Elements* in this axiomatic setting in detail. In Section 1.4 (pp. 67-70) author gives a short presentation of two other contemporary versions of axiomatic systems for Euclidean geometry: Borsuk-Szmielew's and Tarsky's. Finally, the last Section 1.5 (pp. 70-83) contains a detailed exposition of a new, original model of semi-Euclidean plane. The term "semi-Euclidean" belongs to Hartshorne, and denotes plane geometry in which the sum of angles in a triangle is π, yet the parallel postulate fails. Unlike some previous such models, which involved non-Euclidean fields, to build this model author employs Euclidean field (an ordered field for which every non-negative element is a square) – the field of hyperreal numbers. The hyperreals $\mathbb{R}^*$ are introduced using the ultrafilter construction (which depends on the axiom of choice, or at least on the ultrafilter lemma, and the uniqueness of the construction depends on the continuum hypothesis, so one has to work under ZFC+CH). The model of semi-Euclidean plane is a subspace $L \times L$ of the Cartesian plane $\mathbb{R}^* \times \mathbb{R}^*$ over the non-Archimedean field of hyperreal numbers $\mathbb{R}^*$, where L is a subset of hyperreals bounded by some real number. Author proves that it is a semi-euclidean plane, discusses which Euclid's propositions do not hold in this plane, and compares it with Klein's and Poincare's disk models for non-Euclidean geometry.

The second chapter is devoted to derivation of the Thales' theorem within the framework of various axiomatic systems: Hilbert and Hartshorn (2.1), Szmielew and Tarski (2.2) and some other (Borsuk-Szmielew, Birkhoff, Millman-Parker).

The third chapter is about the area method in proving theorems in Euclidean plane geometry. The axiomatic approach follows the work of Janičić, Narboux and Quaresma. Then, in 3.2, the derivation of Thales' theorem in Euclid's Elements is compared with this area approach. The model for this geometry is developed in 3.3.

The fourth chapter contains the exposition of automatic theorem proving based on the area method, and on the work of Janičić and Quaresma. It utilizes the theorem prover GCLC to prove all theorems in Euclid's Book VI (section 4.3).

Chapter five is devoted to foundations of geometry in education. It contains analysis of some contemporary high school textbooks used in Poland and Ukraine, based on the four criteria, formulated on the basis of classical works of Hilbert, Borsuk, Szmielew, Tarski, as well as Hartshorne and Fitzpatrick. In the referee's opinion, the formulated criteria are indeed essential for evaluation and comparison of textbooks. These are first, existence of discussion of primitive concepts, second, existence of discussion of separation in a plane, third, existence of comparison and algeebra of line segments and angles, and fourth, the relation between criteria for congruent triangles and the parallel axiom. Four Polish and three Ukrainian textbooks are compared. In addition, six Polish and four Ukrainian university textbooks are analyzed. All textbooks are compared by their discussion of the Thales' theorem. (5.4).

At last, Chapter six contains some conclusions and concise recommendations for university courses for prospective teachers (6.1) and also some short recommendations for secondary schools with respect to parallel axiom (6.2) and to Thales' theorem (6.3). Recommendations for the secondary school curriculum include the use of application *Euclidea*, the rule that criteria for congruent triangles precede the parallel postulate, and the use of graphic patterns related to Euclid's proposition VI.1 and the co-side theorem.

Conclusion

The manuscript is well-written and covers broad area of axiomatic introduction of Euclidean plane geometry in general and particularly in the high school curricula. This subject is also a priority interest in my personal research. The author has shown a high level of comprehension of the subject. The text also contains some original publishable or published results in the area, such as the model for semi-Euclidean plane, and proper recommendations for improvement of high school geometry curriculum. The manuscript is a valuable contribution to the subject of axiomatic introduction of plane geometry. A small number of misprints or misedited text has been noted, but this does not affect the quality of the work.

To conclude, the referee gladly confirms that the manuscript "Foundations of geometry for secondary schools and prospective teachers" written by Anna Petiurenko is suitable for obtaining a doctoral degree.

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